Integral of a Bivariate Normal Distribution Over a Polygon

Example 9. Let the random variables X and Y have a bivariate normal distribution with parameters $m_x=5$, $\sigma_x=2$, $m_y=9$, $\sigma_y=4$, and $\rho=.5$. Find the probability that the point (X, Y) be inside the triangle whose vertices are $A=(7,8)$, $B=(9,13)$, and $C=(2,9)$.

When obtaining the integral of a bivariate normal distribution over a polygon, it is first necessary to use **26.3.22** in order to transform the variates so that one deals with a circular normal distribution. The polygon in the region of the transformed variables is then divided into configurations such that the integral over any selected configuration can be easily obtained. Below are listed some of the most useful configurations.

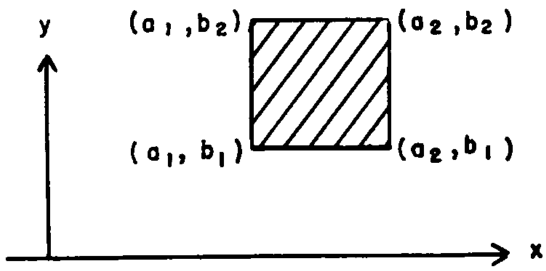

FIGURE 26.5

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2} g(x,y,0)dxdy=[P(a_2)-P(a_1)][P(b_2)-P(b_1)]$$

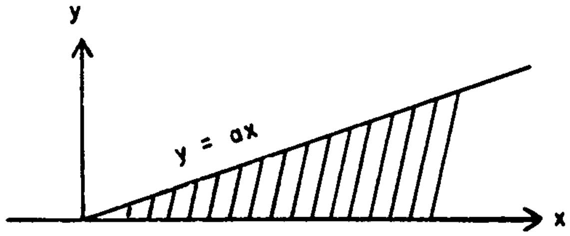

FIGURE 26.6

$$\int_0^\infty\int_0^{ax} g(x,y,0)dxdy=\frac{\arctan a}{2\pi}$$

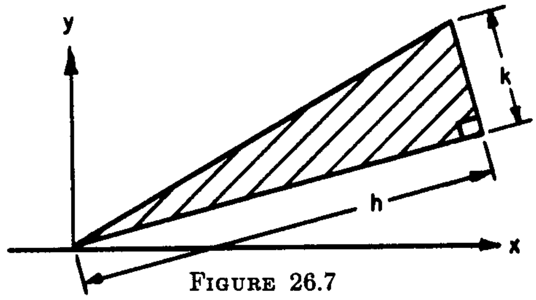

FIGURE 26.7

$$\int_0^h\int_0^{\frac{k}{h}x} g(x,y,0)dxdy=V(h,k)^{11}$$

[11] See **26.3.23** for definition of $V(h, k)$.

For the following two configurations we define

$$h=\frac{|t_2s_1-t_1s_2|}{[(s_2-s_1)^2+(t_2-t_1)^2]^{\frac{1}{2}}}$$

$$k_1=\frac{|s_1(s_2-s_1)+t_1(t_2-t_1)|}{[(s_2-s_1)^2+(t_2-t_1)^2]^{\frac{1}{2}}}$$

$$k_2=\frac{|s_2(s_2-s_1)+t_2(t_2-t_1)|}{[(s_2-s_1)^2+(t_2-t_1)^2]^{\frac{1}{2}}}$$

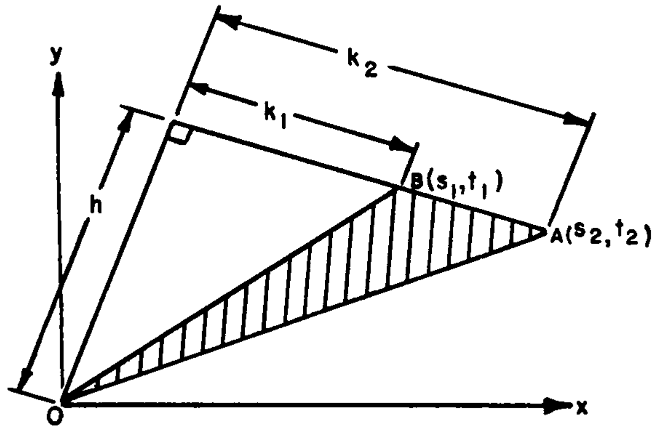

FIGURE 26.8

$$\iint_{\Delta AOB} g(x,y,0)dxdy=V(h,k_2)-V(h,k_1)$$

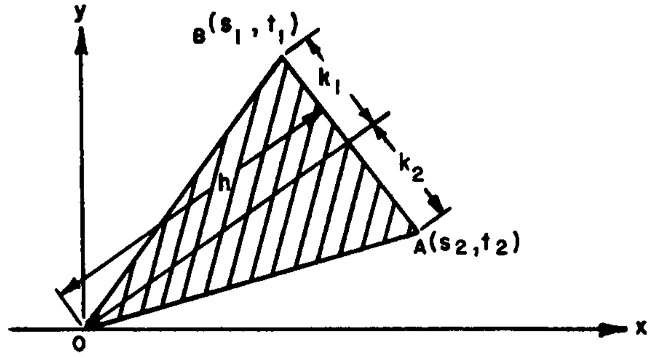

FIGURE 26.9

$$\iint_{\Delta AOB} g(x,y,0)dxdy=V(h,k_2)+V(h,k_1)$$

Using the circularizing transformation **26.3.22** for our example results in

$$s=\frac{1}{\sqrt{3}}\left(\frac{x-5}{2}+\frac{y-9}{4}\right)$$

$$t=-\frac{1}{1}\left(\frac{x-5}{2}-\frac{y-9}{4}\right)$$