26.6. F-(Variance-Ratio) Distribution Function

26.6.1

$$P(F|\nu_1,\nu_2)=\frac{\nu_1^{\frac{1}{2}\nu_1}\nu_2^{\frac{1}{2}\nu_2}}{B\left(\frac{1}{2}\nu_1,\frac{1}{2}\nu_2\right)}\int_0^F t^{\frac{1}{2}(\nu_1-2)}(\nu_2+\nu_1 t)^{-\frac{1}{2}(\nu_1+\nu_2)}dt \qquad (F\geq 0)$$

26.6.2

$$Q(F|\nu_1,\nu_2)=1-P(F|\nu_1,\nu_2)=I_x\left(\frac{\nu_2}{2},\frac{\nu_1}{2}\right)$$

where

$$x=\frac{\nu_2}{\nu_2+\nu_1 F}$$

Relation to the Chi-Square Distribution

If X_1^2 and X_2^2 are independent random variables following chi-square distributions **26.4.1** with ν_1 and ν_2 degrees of freedom respectively, then the distribution of $F=\frac{X_1^2/\nu_1}{X_2^2/\nu_2}$ is said to follow the variance ratio or F-distribution with ν_1 and ν_2 degrees of freedom. The corresponding distribution function is $P(F|\nu_1,\nu_2)$.

Statistical Properties

26.6.3

mean: $$m=\frac{\nu_2}{\nu_2-2} \qquad (\nu_2>2)$$

variance: $$\sigma^2=\frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)} \qquad (\nu_2>4)$$

third central moment:

$$\mu_3=\left(\frac{\nu_2}{\nu_1}\right)^3\frac{8\nu_1(\nu_1+\nu_2-2)(2\nu_1+\nu_2-2)}{(\nu_2-2)^3(\nu_2-4)(\nu_2-6)} \qquad (\nu_2>6)$$

moments about the origin:

$$\mu_n'=\left(\frac{\nu_2}{\nu_1}\right)^n\frac{\Gamma\left(\frac{\nu_1+2n}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{\Gamma\left(\frac{\nu_1-2n}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)} \qquad (\nu_2>2n)$$

characteristic function:

$$\phi(t)=E(e^{iFt})=M\left(\frac{\nu_1}{2},-\frac{\nu_2}{2},-\frac{\nu_2}{\nu_1}it\right)$$

Series Expansions

$$x=\frac{\nu_2}{\nu_2+\nu_1 F}$$

26.6.4

* $$Q(F|\nu_1,\nu_2)=x^{\nu_2/2}\left[1+\frac{\nu_2}{2}(1-x)+\frac{\nu_2(\nu_2+2)}{2\cdot 4}(1-x)^2+\ldots+\frac{\nu_2(\nu_2+2)\ldots(\nu_2+\nu_1-4)}{2\cdot 4\ldots(\nu_1-2)}(1-x)^{\frac{\nu_1-2}{2}}\right] \qquad (\nu_1 \text{ even})$$

*See page II.

26.6.5

$$Q(F|\nu_1,\nu_2)=1-(1-x)^{\nu_1/2}\left[1+\frac{\nu_1}{2}x+\frac{\nu_1(\nu_1+2)}{2\cdot 4}x^2+\ldots+\frac{\nu_1(\nu_1+2)\ldots(\nu_2+\nu_1-4)}{2\cdot 4\ldots(\nu_2-2)}x^{\frac{\nu_2-2}{2}}\right] \qquad (\nu_2 \text{ even})$$

26.6.6

$$Q(F|\nu_1,\nu_2)=x^{\frac{\nu_1+\nu_2-2}{2}}\left[1+\frac{\nu_1+\nu_2-2}{2}\left(\frac{1-x}{x}\right)+\frac{(\nu_1+\nu_2-2)(\nu_1+\nu_2-4)}{2\cdot 4}\left(\frac{1-x}{x}\right)^2+\ldots+\frac{(\nu_1+\nu_2-2)\ldots(\nu_2+2)}{2\cdot 4\ldots(\nu_1-2)}\left(\frac{1-x}{x}\right)^{\frac{\nu_1-2}{2}}\right] \qquad (\nu_1 \text{ even})$$

26.6.7

$$Q(F|\nu_1,\nu_2)=1-(1-x)^{\frac{\nu_1+\nu_2-2}{2}}\left[1+\frac{\nu_1+\nu_2-2}{2}\left(\frac{x}{1-x}\right)+\ldots+\frac{(\nu_1+\nu_2-2)\ldots(\nu_1+2)}{2\cdot 4\ldots(\nu_2-2)}\left(\frac{x}{1-x}\right)^{\frac{\nu_2-2}{2}}\right] \qquad (\nu_2 \text{ even})$$

26.6.8

$$Q(F|\nu_1,\nu_2)=1-A(t|\nu_2)+\beta(\nu_1,\nu_2) \qquad (\nu_1,\nu_2 \text{ odd})$$

$$A(t|\nu_2)=\begin{cases}\frac{2}{\pi}\left\{\theta+\sin\theta\left[\cos\theta+\frac{2}{3}\cos^3\theta+\ldots+\frac{2\cdot 4\ldots(\nu_2-3)}{3.5\ldots(\nu_2-2)}\cos^{\nu_2-2}\theta\right]\right\} & \text{for } \nu_2>1\\ \frac{2\theta}{\pi} \text{ for } \nu_2=1\end{cases}$$

$$\beta(\nu_1,\nu_2)=\begin{cases}\frac{2}{\sqrt{\pi}}\frac{\left(\frac{\nu_2-1}{2}\right)!}{\left(\frac{\nu_2-2}{2}\right)!}\sin\theta\cos^{\nu_2}\theta\left\{1+\frac{\nu_2+1}{3}\sin^2\theta+\cdots+\frac{(\nu_2+1)(\nu_2+3)\ldots(\nu_1+\nu_2-4)}{3\cdot 5\ldots(\nu_1-2)}\sin^{\nu_1-3}\theta\right\} & \text{for } \nu_2>1\\ 0 \text{ for } \nu_1=1 \quad * \end{cases}$$

where

$$\theta=\arctan\sqrt{\frac{\nu_1}{\nu_2}F}$$

Reflexive Relation

If $F_p(\nu_1,\nu_2)$ and $F_{1-p}(\nu_2,\nu_1)$ satisfy

$$Q(F_p(\nu_1,\nu_2)|\nu_1,\nu_2)=p$$

$$Q(F_{1-p}(\nu_2,\nu_1)|\nu_2,\nu_1)=1-p$$