4.6.7 $\operatorname{arctanh} z = \operatorname{arccoth} z \pm \frac{1}{2}\pi i$

(see **4.5.60**) (according as $\mathscr{I}z \gtreqless 0$)

Fundamental Property

The general solutions of the equations

$$z = \sinh t$$
$$z = \cosh t$$
$$z = \tanh t$$

are respectively

4.6.8 $t = \operatorname{Arcsinh} z = (-1)^k \operatorname{arcsinh} z + k\pi i$

4.6.9 $t = \operatorname{Arccosh} z = \pm \operatorname{arccosh} z + 2k\pi i$

4.6.10 $t = \operatorname{Arctanh} z = \operatorname{arctanh} z + k\pi i$ (k, integer)

Functions of Negative Arguments

4.6.11 $\operatorname{arcsinh}(-z) = -\operatorname{arcsinh} z$

***4.6.12** $\operatorname{arccosh}(-z) = \pi i - \operatorname{arccosh} z$

4.6.13 $\operatorname{arctanh}(-z) = -\operatorname{arctanh} z$

Relation to Inverse Circular Functions (see 4.4.20 to 4.4.25)

Hyperbolic identities can be derived from trigonometric identities by replacing z by iz.

4.6.14 $\operatorname{Arcsinh} z = -i \operatorname{Arcsin} iz$

4.6.15 $\operatorname{Arccosh} z = \pm i \operatorname{Arccos} z$

4.6.16 $\operatorname{Arctanh} z = -i \operatorname{Arctan} iz$

4.6.17 $\operatorname{Arccsch} z = i \operatorname{Arccsc} iz$

4.6.18 $\operatorname{Arcsech} z = \pm i \operatorname{Arcsec} z$

4.6.19 $\operatorname{Arccoth} z = i \operatorname{Arccot} iz$

Logarithmic Representations

4.6.20 $\operatorname{arcsinh} x = \ln [x + (x^2+1)^{\frac{1}{2}}]$

4.6.21 $\operatorname{arccosh} x = \ln [x + (x^2-1)^{\frac{1}{2}}]$ $(x \geq 1)$

4.6.22 $\operatorname{arctanh} x = \frac{1}{2} \ln \frac{1+x}{1-x}$ $(0 \leq x^2 < 1)$

4.6.23 $\operatorname{arccsch} x = \ln \left[\frac{1}{x} + \left(\frac{1}{x^2}+1\right)^{\frac{1}{2}}\right]$ $(x \neq 0)$

4.6.24 $\operatorname{arcsech} x = \ln \left[\frac{1}{x} + \left(\frac{1}{x^2}-1\right)^{\frac{1}{2}}\right]$ $(0 < x \leq 1)$

4.6.25 $\operatorname{arccoth} x = \frac{1}{2} \ln \frac{x+1}{x-1}$ $(x^2 > 1)$

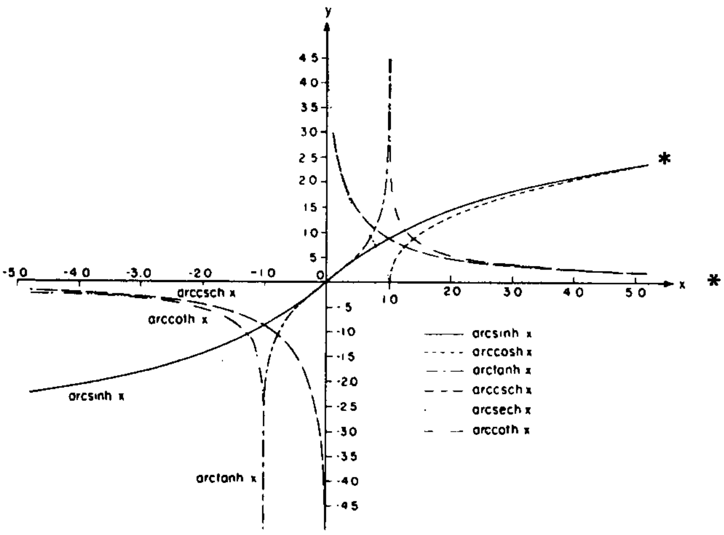

FIGURE 4.8. *Inverse hyperbolic functions.*

Addition and Subtraction of Two Inverse Hyperbolic Functions

4.6.26

$$\operatorname{Arcsinh} z_1 \pm \operatorname{Arcsinh} z_2 = \operatorname{Arcsinh} [z_1(1+z_2^2)^{\frac{1}{2}} \pm z_2(1+z_1^2)^{\frac{1}{2}}]$$

4.6.27

$$\operatorname{Arccosh} z_1 \pm \operatorname{Arccosh} z_2 = \operatorname{Arccosh} \{z_1 z_2 \pm [(z_1^2-1)(z_2^2-1)]^{\frac{1}{2}}\}$$

4.6.28

$$\operatorname{Arctanh} z_1 \pm \operatorname{Arctanh} z_2 = \operatorname{Arctanh} \left(\frac{z_1 \pm z_2}{1 \pm z_1 z_2}\right)$$

4.6.29

$$\operatorname{Arcsinh} z_1 \pm \operatorname{Arccosh} z_2 = \operatorname{Arcsinh}\{z_1 z_2 \pm [(1+z_1^2)(z_2^2-1)]^{\frac{1}{2}}\}$$
$$= \operatorname{Arccosh} [z_2(1+z_1^2)^{\frac{1}{2}} \pm z_1(z_2^2-1)^{\frac{1}{2}}]$$

4.6.30

$$\operatorname{Arctanh} z_1 \pm \operatorname{Arccoth} z_2 = \operatorname{Arctanh} \left(\frac{z_1 z_2 \pm 1}{z_2 \pm z_1}\right)$$
$$= \operatorname{Arccoth} \left(\frac{z_2 \pm z_1}{z_1 z_2 \pm 1}\right)$$

*See page II.