24. Combinatorial Analysis

Mathematical Properties

In each sub-section of this chapter we use a fixed format which emphasizes the use and methods of extending the accompanying tables. The format follows this form:

I. Definitions

A. Combinatorial
B. Generating functions
C. Closed form

II. Relations

A. Recurrences
B. Checks in computing
C. Basic use in numerical analysis

III. Asymptotic and Special Values

In general the notations used are standard. This includes the difference operator Δ defined on functions of x by $\Delta f(x)=f(x+1)-f(x)$, $\Delta^{n+1}f(x)=\Delta(\Delta^n f(x))$, the Kronecker delta δ_{ij}, the Riemann zeta function $\zeta(s)$ and the greatest common divisor symbol (m, n). The range of the summands for a summation sign without limits is explained to the right of the formula.

The notations which are not standard are those for the multinomials which are arbitrary shorthand for use in this chapter, and those for the Stirling numbers which have never been standardized. A short table of various notations for these numbers follows:

Notations for the Stirling Numbers

Reference	*First Kind*	*Second Kind*
This chapter	$S_n^{(m)}$	$\mathfrak{S}_n^{(m)}$
[24.2] Fort	$S_n^{(m)}$	$\mathscr{S}_n^{(m)}$ *
[24.7] Jordan	S_n^m	$\mathfrak{S}_n^m$ *
[24.10] Moser and Wyman	S_n^m	σ_n^m
[24.9] Milne-Thomson	$\binom{n-1}{m-1} B_{n-m}^{(n)}$	$\binom{n}{m} B_{n-m}^{(-m)}$
[24.15] Riordan	$s(n, m)$	$S(n, m)$
[24.1] Carlitz, [24.3] Gould	$(-1)^{n-m}S_1(n-1, n-m)$	$S_2(m, n-m)$
Miksa (Unpublished tables)	$S(n-m+1, n)$	${}_mS_n$
[24.17] Gupta		$u(n, m)$

We feel that a capital S is natural for Stirling numbers of the first kind; it is infrequently used for other notation in this context. But once it is used we have difficulty finding a suitable symbol for Stirling numbers of the second kind. The numbers are sufficiently important to warrant a special and easily recognizable symbol, and yet that symbol must be easy to write. We have settled on a script capital $\mathfrak{S}$ without any certainty that we have settled this question permanently.

We feel that the subscript-superscript notation emphasizes the generating functions (which are powers of mutually inverse functions) from which most of the important relations flow.

24.1. Basic Numbers

24.1.1 Binomial Coefficients

I. Definitions

A. $\binom{n}{m}$ is the number of ways of choosing m objects from a collection of n distinct objects without regard to order.

B. Generating functions

$$* \quad (1+x)^n=\sum_{m=0}^{n}\binom{n}{m}x^m \qquad n=0, 1, \ldots$$

$$(1-x)^{-m-1}=\sum_{n=m}^{\infty}\binom{n}{m}x^{n-m} \qquad |x|<1$$

C. Closed form

$$\binom{n}{m}=\frac{n!}{m!(n-m)!}=\binom{n}{n-m}$$

$$=\frac{n(n-1)\ldots(n-m+1)}{m!} \qquad n\geq m$$

II. Relations

A. Recurrences

$$\binom{n+1}{m}=\binom{n}{m}+\binom{n}{m-1} \qquad n\geq m\geq 1$$

$$=\binom{n}{m}+\binom{n-1}{m-1}+\ldots+\binom{n-m}{0} \qquad n\geq m$$

B. Checks

$$\sum_{m=0}^{n}\binom{r}{m}\binom{s}{n-m}=\binom{r+s}{n} \qquad r+s\geq n$$

$$\sum_{m=0}^{n}(-1)^{n-m}\binom{r}{m}=\binom{r-1}{n} \qquad r\geq n+1$$

$$\binom{n}{m}\equiv\binom{n_0}{m_0}\binom{n_1}{m_1}\ldots \pmod{p} \qquad p \text{ a prime}$$

*See page II.