23.1.22 $B_n(\tfrac{1}{4})=(-1)^n B_n(\tfrac{3}{4})$
$$=-2^{-n}(1-2^{1-n})B_n-n4^{-n}E_{n-1} \qquad n=1,2,\ldots$$

23.1.23 $B_{2n}(\tfrac{1}{3})=B_{2n}(\tfrac{2}{3})$
$$=-2^{-1}(1-3^{1-2n})B_{2n} \qquad n=0,1,\ldots$$

23.1.24 $B_{2n}(\tfrac{1}{6})=B_{2n}(\tfrac{5}{6})$
$$=2^{-1}(1-2^{1-2n})(1-3^{1-2n})B_{2n} \qquad n=0,1,\ldots$$

$E_{2n-1}(\tfrac{1}{3})=-E_{2n-1}(\tfrac{2}{3})$
$$=-(2n)^{-1}(1-3^{1-2n})(2^{2n}-1)B_{2n} \qquad n=1,2,\ldots$$

Symbolic Operations

23.1.25 $p(B(x)+1)-p(B(x))=p'(x)$ $\qquad$ $p(E(x)+1)+p(E(x))=2p(x)$

23.1.26 $B_n(x+h)=(B(x)+h)^n \qquad n=0,1,\ldots$ $\qquad$ $E_n(x+h)=(E(x)+h)^n \qquad n=0,1,\ldots$

Here $p(x)$ denotes a polynomial in x and after expanding we set $\{B(x)\}^n=B_n(x)$ and $\{E(x)\}^n=E_n(x)$.

Relations Between the Polynomials

23.1.27
$$E_{n-1}(x)=\frac{2^n}{n}\left\{B_n\left(\frac{x+1}{2}\right)-B_n\left(\frac{x}{2}\right)\right\}$$
$$=\frac{2}{n}\left\{B_n(x)-2^nB_n\left(\frac{x}{2}\right)\right\} \qquad n=1,2,\ldots$$

23.1.28
$$E_{n-2}(x)=2\binom{n}{2}^{-1}\sum_{k=0}^{n-2}\binom{n}{k}(2^{n-k}-1)B_{n-k}B_k(x) \qquad n=2,3,\ldots$$

23.1.29
$$B_n(x)=2^{-n}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}E_k(2x) \qquad n=0,1,\ldots$$

Euler-Maclaurin Formulas

Let $F(x)$ have its first $2n$ derivatives continuous on an interval (a, b). Divide the interval into m equal parts and let $h=(b-a)/m$. Then for some θ, $1>\theta>0$, depending on $F^{(2n)}(x)$ on (a, b), we have

23.1.30
$$\sum_{k=0}^{m}F(a+kh)=\frac{1}{h}\int_a^b F(t)dt+\frac{1}{2}\{F(b)+F(a)\}$$
$$+\sum_{k=1}^{n-1}\frac{h^{2k-1}}{(2k)!}B_{2k}\{F^{(2k-1)}(b)-F^{(2k-1)}(a)\}$$
$$+\frac{h^{2n}}{(2n)!}B_{2n}\sum_{k=0}^{m-1}F^{(2n)}(a+kh+\theta h)$$

Equivalent to this is

23.1.31
$$\frac{1}{h}\int_x^{x+h}F(t)dt=\frac{1}{2}\{F(x+h)+F(x)\}$$
$$-\sum_{k=1}^{n-1}\frac{h^{2k-1}}{(2k)!}B_{2k}\{F^{(2k-1)}(x+h)-F^{(2k-1)}(x)\}$$
$$-\frac{h^{2n}}{(2n)!}B_{2n}F^{(2n)}(x+\theta h) \qquad b-h\geq x\geq a$$

Let $\hat{B}_n(x)=B_n(x-[x])$. The Euler Summation Formula is

23.1.32
$$\sum_{k=0}^{m-1}F(a+kh+\omega h)=\frac{1}{h}\int_a^b F(t)dt$$
$$+\sum_{k=1}^{p}\frac{h^{k-1}}{k!}B_k(\omega)\{F^{(k-1)}(b)-F^{(k-1)}(a)\}$$
$$-\frac{h^p}{p!}\int_0^1\hat{B}_p(\omega-t)\left\{\sum_{k=0}^{m-1}F^{(p)}(a+kh+th)\right\}dt$$
$$p\leq 2n,\ 1\geq\omega\geq 0$$