### Integrals

**23.1.11** $$\int_a^x B_n(t)dt=\frac{B_{n+1}(x)-B_{n+1}(a)}{n+1}$$

$$\int_a^x E_n(t)dt=\frac{E_{n+1}(x)-E_{n+1}(a)}{n+1}$$

**23.1.12** $$\int_0^1 B_n(t)B_m(t)dt=(-1)^{n-1}\frac{m!n!}{(m+n)!}B_{m+n} \qquad m,n=1,2,\ldots$$

$$\int_0^1 E_n(t)E_m(t)dt=(-1)^n 4(2^{m+n+2}-1)\frac{m!n!}{(m+n+2)!}B_{m+n+2} \qquad m,n=0,1,\ldots$$

(The polynomials are orthogonal for $m+n$ odd.)

### Inequalities

**23.1.13** $$|B_{2n}|>|B_{2n}(x)| \qquad n=1,2,\ldots, \quad 1>x>0$$

$$4^{-n}|E_{2n}|>(-1)^n E_{2n}(x)>0 \qquad n=1,2,\ldots, \quad \tfrac{1}{2}>x>0$$

**23.1.14**

$$\frac{2(2n+1)!}{(2\pi)^{2n+1}}\left(\frac{1}{1-2^{-2n}}\right)>(-1)^{n+1}B_{2n+1}(x)>0 \qquad n=1,2,\ldots, \quad \tfrac{1}{2}>x>0$$

$$\frac{4(2n-1)!}{\pi^{2n}}\left(1+\frac{1}{2^{2n}-2}\right)>(-1)^n E_{2n-1}(x)>0 \qquad n=1,2,\ldots, \quad \tfrac{1}{2}>x>0$$

**23.1.15**

$$\frac{2(2n)!}{(2\pi)^{2n}}\left(\frac{1}{1-2^{1-2n}}\right)>(-1)^{n+1}B_{2n}>\frac{2(2n)!}{(2\pi)^{2n}} \qquad n=1,2,\ldots$$

$$\frac{4^{n+1}(2n)!}{\pi^{2n+1}}>(-1)^n E_{2n}>\frac{4^{n+1}(2n)!}{\pi^{2n+1}}\left(\frac{1}{1+3^{-1-2n}}\right) \qquad n=0,1,\ldots$$

### Fourier Expansions

**23.1.16**

$$B_n(x)=-2\frac{n!}{(2\pi)^n}\sum_{k=1}^{\infty}\frac{\cos(2\pi kx-\frac{1}{2}\pi n)}{k^n} \qquad n>1, 1\geq x\geq 0; \quad n=1, 1>x>0$$

$$E_n(x)=4\frac{n!}{\pi^{n+1}}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x-\frac{1}{2}\pi n)}{(2k+1)^{n+1}} \qquad n>0, 1\geq x\geq 0; \quad n=0, 1>x>0$$

**23.1.17**

$$B_{2n-1}(x)=\frac{(-1)^n 2(2n-1)!}{(2\pi)^{2n-1}}\sum_{k=1}^{\infty}\frac{\sin 2k\pi x}{k^{2n-1}} \qquad n>1, 1\geq x\geq 0; \quad n=1, 1>x>0$$

$$E_{2n-1}(x)=\frac{(-1)^n 4(2n-1)!}{\pi^{2n}}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi x}{(2k+1)^{2n}} \qquad n=1,2,\ldots, \quad 1\geq x\geq 0$$

**23.1.18**

$$B_{2n}(x)=\frac{(-1)^{n-1}2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos 2k\pi x}{k^{2n}} \qquad n=1,2,\ldots, \quad 1\geq x\geq 0$$

$$E_{2n}(x)=\frac{(-1)^n 4(2n)!}{\pi^{2n+1}}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\pi x}{(2k+1)^{2n+1}} \qquad n>0, 1\geq x\geq 0; \quad n=0, 1>x>0$$

### Special Values

**23.1.19** $$B_{2n+1}=0 \qquad n=1,2,\ldots$$

$$E_{2n+1}=0 \qquad n=0,1,\ldots$$

**23.1.20** $$B_n(0)=(-1)^n B_n(1)=B_n \qquad n=0,1,\ldots$$

$$E_n(0)=-E_n(1)=-2(n+1)^{-1}(2^{n+1}-1)B_{n+1} \qquad n=1,2,\ldots$$

**23.1.21** $$B_n(\tfrac{1}{2})=-(1-2^{1-n})B_n \qquad n=0,1,\ldots$$

$$E_n(\tfrac{1}{2})=2^{-n}E_n \qquad n=0,1,\ldots$$