Change of Interval of Orthogonality

In some applications it is more convenient to use polynomials orthogonal on the interval [0, 1]. One can obtain the new polynomials from the ones given in this chapter by the substitution $x=2\bar{x}-1$. The coefficients of the new polynomial can be computed from the old by the following recursive scheme, provided the standardization is not changed. If

$$f_n(x)=\sum_{m=0}^{n} a_m x^m, \quad f_n^*(x)=f_n(2x-1)=\sum_{m=0}^{n} a_m^* x^m$$

then the a_m^* are given recursively by the a_m through the relations

$a_m^{(j)}=2a_m^{(j-1)}-a_{m+1}^{(j)}; \ m=n-1, n-2, \ldots, j; \ j=0, 1, 2, \ldots, n$

$a_m^{(-1)}=a_m/2, \ m=0, 1, 2, \ldots, n$

$a_n^{(j)}=2^j a_n, \ j=0, 1, 2, \ldots, n$ and $a_m^{(m)}=a_m^*; \ m=0, 1, 2, \ldots, n.$

Example 4. Given $T_5(x)=5x-20x^3+16x^5$, find $T_5^*(x)$.

$j \backslash m$	5	4	3	2	1	0
−1	$8=a_5^{(-1)}$	0	$-10=a_3^{(-1)}$	0	$2.5=a_1^{(-1)}$	0
0	16	−16	−4	4	1	$-1=a_0^*$
1	32	−64	56	−48	$50=a_1^*$	
2	64	−192	304	$-400=a_2^*$		
3	128	−512	$1120=a_3^*$			
4	256	$-1280=a_4^*$				
5	$512=a_5^*$					

Hence, $T_5^*(x)=512x^5-1280x^4+1120x^3-400x^2+50x-1$.

22.19. Least Square Approximations

Problem: Given a function $f(x)$ (analytically or in form of a table) in a domain D (which may be a continuous interval or a set of discrete points).[2] Approximate $f(x)$ by a polynomial $F_n(x)$ of given degree n such that a weighted sum of the squares of the errors in D is least.

Solution: Let $w(x)\geq 0$ be the weight function chosen according to the relative importance of the errors in different parts of D. Let $f_m(x)$ be orthogonal polynomials in D relative to $w(x)$, i.e. $(f_m, f_n)=0$ for $m\neq n$, where

$$(f,g)=\begin{cases} \int_D w(x)f(x)g(x)dx & \text{if } D \text{ is a continuous interval} \\ \sum_{m=1}^{N} w(x_m)f(x_m)g(x_m) & \text{if } D \text{ is a set of } N \text{ discrete points } x_m. \end{cases}$$

Then

$$F_n(x)=\sum_{m=0}^{n} a_m f_m(x)$$

where

* $$a_m=(f,f_m)/(f_m,f_m).$$

[2] $f(x)$ has to be square integrable, see e.g. [22.17].

*See page II.

D a Continuous Interval

Example 5. Find a least square polynomial of degree 5 for $f(x)=\frac{1}{1+x}$, in the interval $2\leq x\leq 5$, using the weight function

$$w(x)=\frac{1}{\sqrt{(x-2)(5-x)}}$$

which stresses the importance of the errors at the ends of the interval.

Reduction to interval $[-1,1]$, $t=\frac{2x-7}{3}$

$$w(x(t))=\frac{2}{3}\frac{1}{\sqrt{1-t^2}}$$

From 22.2, $f_m(t)=T_m(t)$ and

$$a_m=\frac{4}{3\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\frac{1}{t+3}T_m(t)dt \qquad (m\neq 0)$$

$$a_0=\frac{2}{3\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\frac{dt}{t+3}$$