

22.2. Orthogonality Relations

	$f_n(x)$	Name of Polynomial	a	b	$w(x)$	Standardization	h_n	Remarks
22.2.1	$P_n^{(\alpha, \beta)}(x)$	Jacobi	-1	1	$(1-x)^\alpha(1+x)^\beta$	$P_n^{(\alpha, \beta)}(1) = \binom{n+\alpha}{n}$	$\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}$	$\alpha > -1, \beta > -1$
22.2.2	$G_n(p, q, x)$	Jacobi	0	1	$(1-x)^p x^q$	$k_n = 1$	$\frac{n!\Gamma(n+q)\Gamma(n+p)\Gamma(n+p-q+1)}{(2n+p)\Gamma^2(2n+p)}$	$p-q > -1, q > 0$
22.2.3	$C_n^{(\alpha)}(x)$	Ultraspherical (Gegenbauer)	-1	1	$(1-x^2)^{\alpha-\frac{1}{2}}$	$C_n^{(\alpha)}(1) = \binom{n+2\alpha-1}{n}$ ($\alpha \neq 0$)	$\frac{\pi 2^{1-2\alpha} \Gamma(n+2\alpha)}{n!(n+\alpha)[\Gamma(\alpha)]^2} \quad \alpha \neq 0$	$\alpha > -\frac{1}{2}$
						$C_n^{(0)}(1) = \frac{2}{n}$ $C_0^{(0)}(1) = 1$	$\frac{2\pi}{n^2} \quad \alpha = 0$	
22.2.4	$T_n(x)$	Chebyshev of the first kind	-1	1	$(1-x^2)^{-\frac{1}{2}}$	$T_n(1) = 1$	$\begin{cases} \frac{\pi}{2} & n \neq 0 \\ \pi & n = 0 \end{cases}$	
22.2.5	$U_n(x)$	Chebyshev of the second kind	-1	1	$(1-x^2)^{\frac{1}{2}}$	$U_n(1) = n+1$	$\frac{\pi}{2}$	8
* 22.2.6	$C_n(x)$	Chebyshev of the first kind	-2	2	$\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}$	$C_n(2) = 2$	$\begin{cases} 4\pi & n \neq 0 \\ 8\pi & n = 0 \end{cases}$	
* 22.2.7	$S_n(x)$	Chebyshev of the second kind	-2	2	$\left(1-\frac{x^2}{4}\right)^{\frac{1}{2}}$	$S_n(2) = n+1$	π	
22.2.8	$T_n^*(x)$	Shifted Chebyshev of the first kind	0	1	$(x-x^2)^{-\frac{1}{2}}$	$T_n^*(1) = 1$	$\begin{cases} \frac{\pi}{2} & n \neq 0 \\ \pi & n = 0 \end{cases}$	
22.2.9	$U_n^*(x)$	Shifted Chebyshev of the second kind	0	1	$(x-x^2)^{\frac{1}{2}}$	$U_n^*(1) = n+1$	$\frac{\pi}{8} \quad *$	
22.2.10	$P_n(x)$	Legendre (Spherical)	-1	1	1	$P_n(1) = 1$	$\frac{2}{2n+1}$	
22.2.11	$P_n^*(x)$	Shifted Legendre	0	1	1		$\frac{1}{2n+1}$	

*See page II.