**21.7.20**

$$S_{mn}(c,\eta)=\eta(1-\eta^2)^{m/2}\sum_{k=0}^{\infty}c_{2k}^{mn}(1-\eta^2)^k \qquad (n-m)\text{ odd}$$

$$c_{2k}^{mn}=\frac{1}{2^m k!(m+k)!}\sum_{r=k}^{\infty}\frac{(2m+2r)!}{(2r)!}(-r)_k\left(m+r+\frac{1}{2}\right)_k d_{2r}^{mn} \qquad (n-m)\text{ even}$$

$$c_{2k}^{mn}=\frac{1}{2^m k!(m+k)!}\sum_{r=k}^{\infty}\frac{(2m+2r+1)!}{(2r+1)!}(-r)_k\left(m+r+\frac{3}{2}\right)_k d_{2r+1}^{mn} \qquad (n-m)\text{ odd}$$

$$(\alpha)_k=\alpha(\alpha+1)(\alpha+2)\ .\ .\ .\ (\alpha+k+1)$$

(The $d_r^{mn}$'s are the coefficients in **21.7.1.**)

### Prolate Angular Functions—Second Kind

Expansion **21.7.2** ultimately leads to

**21.7.21**

$$S_{mn}^{(2)}(c,\eta)=\sum_{r=-2m,\,-2m+1}^{\infty}{}' d_r^{mn}Q_{m+r}^m(\eta)+\sum_{r=2m+2,\,2m+1}^{\infty}{}' d_{\rho|r}^{mn}P_{r-m-1}^m(\eta)$$

(The coefficients $d_r^{mn}$ are the same as in **21.7.1**; the coefficients $d_{\rho|r}^{mn}$ are tabulated in [21.4].)

## 21.8. Oblate Angular Functions

### Power Series Expansion for Eigenvalues

**21.8.1** $$\lambda_{mn}=\sum_{k=0}^{\infty}(-1)^k l_{2k}c^{2k}$$

where the $l_k$'s are the same as in **21.7.5.**

### Asymptotic Expansion for Eigenvalues [21.4]

**21.8.2**

$$\lambda_{mn}=-c^2+2c(2\nu+m+1)-2\nu(\nu+m+1)-(m+1)+\Lambda_{mn}$$

$$\nu=\frac{1}{2}(n-m)\text{ for }(n-m)\text{ even};\qquad \nu=\frac{1}{2}(n-m-1)\text{ for }(n-m)\text{ odd}$$

$$\Lambda_{mn}=\sum_{k=1}^{\infty}\beta_k^{mn}c^{-k}$$

$$\beta_1^{mn}=-2^{-3}q(q^2+1-m^2)$$

$$\beta_2^{mn}=-2^{-6}[5q^4+10q^2+1-2m^2(3q^2+1)+m^4]$$

$$\beta_3^{mn}=-2^{-9}q[33q^4+114q^2+37-2m^2(23q^2+25)+13m^4]$$

$$\beta_4^{mn}=-2^{-10}[63q^6+340q^4+239q^2+14-10m^2(10q^4+23q^2+3)+m^4(39q^2-18)-2m^6]$$

$$\beta_k^{mn}=\nu(\nu+m)a_k^{-1}+(\nu+1)(\nu+m+1)a_k^{+1}$$

$q=n+1$ for $(n-m)$ even; $q=n$ for $(n-m)$ odd

(For the definition of $a_k^{\pm r}$ see **21.8.3.**)

### Asymptotic Expansion for Oblate Angular Functions

**21.8.3**

$$S_{mn}(-ic,\eta)\sim(1-\eta^2)^{m/2}\sum_{s=-\nu}^{\infty}A_s^{mn}\{e^{-c(1-\eta)}L_{\nu+s}^{(m)}[2c(1-\eta)]+(-1)^{n-m}e^{-c(1+\eta)}L_{\nu+s}^{(m)}[2c(1+\eta)]\}$$

where the $L_\nu^{(m)}(x)$ are Laguerre polynomials (see chapter **22**) and

$$\frac{A_{\pm r}^{mn}}{A_0^{mn}}=\sum_{k=r}^{\infty}a_k^{\pm r}(m,n)c^{-k}$$

(Expressions of $a_k^{\pm r}$ are given in [21.4].)

## 21.9. Radial Spheroidal Wave Functions

**21.9.1**

$$R_{mn}^{(p)}(c,\xi)=\left\{\sum_{r=0,\,1}^{\infty}{}'\frac{(2m+r)!}{r!}d_r^{mn}\right\}^{-1}\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}\cdot\sum_{r=0,\,1}^{\infty}{}' i^{r+m-n}\frac{(2m+r)!}{r!}d_r^{mn}Z_{m+r}^{(p)}(c\xi)^{*}$$

$$Z_n^{(p)}(z)=\sqrt{\frac{\pi}{2z}}J_{n+\frac{1}{2}}(z)\qquad(p=1)$$

$$=\sqrt{\frac{\pi}{2z}}Y_{n+\frac{1}{2}}(z)\qquad(p=2)$$

($J_{n+\frac{1}{2}}(z)$ and $Y_{n+\frac{1}{2}}(z)$ are Bessel functions, order $n+\frac{1}{2}$, of the first and second kind respectively (see chapter **10**).)

**21.9.2** $$R_{mn}^{(3)}(c,\xi)=R_{mn}^{(1)}(c,\xi)+iR_{mn}^{(2)}(c,\xi)$$

**21.9.3** $$R_{mn}^{(4)}(c,\xi)=R_{mn}^{(1)}(c,\xi)-iR_{mn}^{(2)}(c,\xi)$$

### Asymptotic Behavior of $R_{mn}^{(1)}(c,\xi)$ and $R_{mn}^{(2)}(c,\xi)$

**21.9.4** $$R_{mn}^{(1)}(c,\xi)\xrightarrow[c\xi\to\infty]{}\frac{1}{c\xi}\cos\left[c\xi-\tfrac{1}{2}(n+1)\pi\right]$$

**21.9.5** $$R_{mn}^{(2)}(c,\xi)\xrightarrow[c\xi\to\infty]{}\frac{1}{c\xi}\sin\left[c\xi-\tfrac{1}{2}(n+1)\pi\right]$$

*See page II.