Evaluation of Coefficients

Step 1. Calculate N_r^m's from

21.7.8

$$N_{r+2}^m=\gamma_r^m-\lambda_{mn}-\frac{\beta_r^m}{N_r^m}\qquad (r\geq 2)$$

$$N_2^m=\gamma_0^m-\lambda_{mn};\ N_3^m=\gamma_1^m-\lambda_{mn}$$

$$\gamma_r^m=(m+r)(m+r+1)+\frac{1}{2}c^2\left[1-\frac{4m^2-1}{(2m+2r-1)(2m+2r+3)}\right]\ (r\geq 0)$$

Step 2. Calculate ratios $\frac{d_0}{d_{2r}}$ and $\frac{d_1}{d_{2p+1}}$ from

21.7.9 $$\frac{d_0}{d_{2r}}=\left(\frac{d_0}{d_2}\right)\left(\frac{d_2}{d_4}\right)\cdots\left(\frac{d_{2r-2}}{d_{2r}}\right)$$

21.7.10 $$\frac{d_1}{d_{2p+1}}=\left(\frac{d_1}{d_3}\right)\left(\frac{d_3}{d_5}\right)\cdots\left(\frac{d_{2p-1}}{d_{2p+1}}\right)$$

and the formula for N_r^m in **21.7.7.**

The coefficients d_r^{mn} are determined to within the arbitrary factor d_0 for r even and d_1 for r odd. The choice of these factors depends on the normalization scheme adopted.

Normalization of Angular Functions

Meixner-Schäfke Scheme

21.7.11 $$\int_{-1}^{1}[S_{mn}(c,\eta)]^2d\eta=\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}$$

Stratton-Morse-Chu-Little-Corbató Scheme

21.7.12 $$\sum_{r=0,1}^{\infty}{}' \frac{(r+2m)!}{r!}d_r=\frac{(n+m)!}{(n-m)!}$$

(This normalization has the effect that $S_{mn}(c,\eta)\to P_n^m(\eta)$ as $\eta\to 1$.)

Flammer Scheme [21.4]

21.7.13

$$S_{mn}(c,0)=P_n^m(0)=\frac{(-1)^{\frac{n-m}{2}}(n+m)!}{2^n\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}\qquad (n-m)\text{ even}$$

21.7.14

$$S'_{mn}(c,0)=P_n^{m\prime}(0)=\frac{(-1)^{\frac{n-m-1}{2}}(n+m+1)!}{2^n\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!}\qquad (n-m)\text{ odd}$$

The above lead to the following conditions for d_r^{mn}

21.7.15

$$\sum_{r=0}^{\infty}{}'\frac{(-1)^{r/2}(r+2m)!}{2^r\left(\frac{r}{2}\right)!\left(\frac{r+2m}{2}\right)!}d_r^{mn}=\frac{(-1)^{\frac{n-m}{2}}(n+m)!}{2^{n-m}\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}\qquad (n-m)\text{ even}$$

21.7.16

$$\sum_{r=1}^{\infty}{}'\frac{(-1)^{\frac{r-1}{2}}(r+2m+1)!}{2^r\left(\frac{r-1}{2}\right)!\left(\frac{r+2m+1}{2}\right)!}d_r^{mn}=\frac{(-1)^{\frac{n-m-1}{2}}(n+m+1)!}{2^{n-m}\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!}\qquad (n-m)\text{ odd}$$

(The normalization scheme **21.7.13** and **21.7.14** is also used in [21.10].)

Asymptotic Expansions for $S_{mn}(c,\eta)$

21.7.17

$$S_{mn}(c,\eta)=(1-\eta^2)^{\frac{1}{2}}U_{mn}(c,\eta)\qquad (c\to\infty)$$

$$U_{mn}(x)=\sum_{r=-\infty}^{\infty}h_r^lD_{l+r}(x)\qquad l=n-m$$

where the $D_r(x)$'s are the parabolic cylinder functions (see chapter **19**).

$$D_r(x)=(-1)^re^{x^2/4}\frac{d^2}{dx^2}e^{-x^2/2}=2^{-r/2}e^{-x^2/4}H_r\left(\frac{x}{\sqrt{2}}\right)$$

and the $H_r(x)$ are the Hermite polynomials (see chapter **22**). (For tables of $h_{\pm r}^l/h_0^l$ see [21.4].)

Expansion of $S_{mn}(c,\eta)$ in Powers of η

21.7.18

$$S_{mn}(c,\eta)=(1-\eta^2)^{m/2}\sum_{r=0,1}^{\infty}{}'p_r^{mn}(c)\eta^r$$

$$(r+1)(r+2)p_{r+2}^{mn}(c)-[r(r+2m+1)+m(m+1)-\lambda_{mn}(c)]p_r^{mn}(c)-c^2p_{r-2}^{mn}(c)=0$$

(The derivation of the transcendental equation for λ_{mn} is similar to the derivation of **21.7.4** from **21.7.3.**)

Expansion of $S_{mn}(c,\eta)$ in Powers of $(1-\eta^2)$

21.7.19

$$S_{mn}(c,\eta)=(1-\eta^2)^{m/2}\sum_{k=0}^{\infty}{}'c_{2k}^{mn}(1-\eta^2)^k\qquad (n-m)\text{ even}$$