Power Series Expansion for λ_{mn}

21.7.5

$$\lambda_{mn}=\sum_{k=0}^{\infty} l_{2k}c^{2k}$$

$$l_0=n(n+1)$$

$$l_2=\tfrac{1}{2}\left[1-\frac{(2m-1)(2m+1)}{(2n-1)(2n+3)}\right]$$

$$l_4=\frac{-(n-m+1)(n-m+2)(n+m+1)(n+m+2)}{2(2n+1)(2n+3)^3(2n+5)}+\frac{(n-m-1)(n-m)(n+m-1)(n+m)}{2(2n-3)(2n-1)^3(2n+1)}$$

$$l_6=(4m^2-1)\left[\frac{(n-m+1)(n-m+2)(n+m+1)(n+m+2)}{(2n-1)(2n+1)(2n+3)^5(2n+5)(2n+7)}-\frac{(n-m-1)(n-m)(n+m-1)(n+m)}{(2n-5)(2n-3)(2n-1)^5(2n+1)(2n+3)}\right]$$

$$l_8=2(4m^2-1)^2A+\frac{1}{16}B+\frac{1}{8}C+\frac{1}{2}D$$

$$A=\frac{(n-m-1)(n-m)(n+m-1)(n+m)}{(2n-5)^2(2n-3)(2n-1)^7(2n+1)(2n+3)^2}-\frac{(n-m+1)(n-m+2)(n+m+1)(n+m+2)}{(2n-1)^2(2n+1)(2n+3)^7(2n+5)(2n+7)^2}$$

$$B=\frac{(n-m-3)(n-m-2)(n-m-1)(n-m)(n+m-3)(n+m-2)(n+m-1)(n+m)}{(2n-7)(2n-5)^2(2n-3)^3(2n-1)^4(2n+1)}$$
$$-\frac{(n-m+1)(n-m+2)(n-m+3)(n-m+4)(n+m+1)(n+m+2)(n+m+3)(n+m+4)}{(2n+1)(2n+3)^4(2n+5)^3(2n+7)^2(2n+9)}$$

$$C=\frac{(n-m+1)^2(n-m+2)^2(n+m+1)^2(n+m+2)^2}{(2n+1)^2(2n+3)^7(2n+5)^2}-\frac{(n-m-1)^2(n-m)^2(n+m-1)^2(n+m)^2}{(2n-3)^2(2n-1)^7(2n+1)^2}$$

$$D=\frac{(n-m-1)(n-m)(n-m+1)(n-m+2)(n+m-1)(n+m)(n+m+1)(n+m+2)}{(2n-3)(2n-1)^4(2n+1)^2(2n+3)^4(2n+5)}$$

Asymptotic Expansion for λ_{mn}

21.7.6

$$\lambda_{mn}(c)=cq+m^2-\frac{1}{8}(q^2+5)-\frac{q}{64c}(q^2+11-32m^2)$$
$$-\frac{1}{1024c^2}[5(q^4+26q^2+21)-384m^2(q^2+1)]$$
$$-\frac{1}{c^3}\left[\frac{1}{128^2}(33q^5+1594q^3+5621q)-\frac{m^2}{128}(37q^3+167q)+\frac{m^4}{8}q\right]$$
$$-\frac{1}{c^4}\left[\frac{1}{256^2}(63q^6+4940q^4+43327q^2+22470)-\frac{m^2}{512}(115q^4+1310q^2+735)+\frac{3m^4}{8}(q^2+1)\right]$$
$$-\frac{1}{c^5}\left[\frac{1}{1024^2}(527q^7+61529q^5+1043961q^3+2241599q)-\frac{m^2}{32\cdot 1024}(5739q^5+127550q^3+298951q)+\frac{m^4}{512}(355q^3+1505q)-\frac{m^6q}{16}\right]+O(c^{-6})$$
$$q=2(n-m)+1$$

Refinement of Approximate Values of λ_{mn}

If $\lambda_{mn}^{(1)}$ is an approximation to λ_{mn} obtained either from **21.7.5** or **21.7.6** then

21.7.7

$$\lambda_{mn}=\lambda_{mn}^{(1)}+\delta\lambda_{mn}$$

$$\delta\lambda_{mn}=\frac{U_1(\lambda_{mn}^{(1)})+U_2(\lambda_{mn}^{(1)})}{\Delta_1+\Delta_2}$$

$$\Delta_1=1+\frac{\beta_r^m}{(N_r^m)^2}+\frac{\beta_r^m\beta_{r-2}^m}{(N_r^mN_{r-2}^m)^2}+\frac{\beta_r^m\beta_{r-2}^m\beta_{r-4}^m}{(N_r^mN_{r-2}^mN_{r-4}^m)^2}+\dots$$

$$\Delta_2=\frac{(N_{r+2}^m)^2}{\beta_{r+2}^m}+\frac{(N_{r+2}^mN_{r+4}^m)^2}{\beta_{r+2}^m\beta_{r+4}^m}+\frac{(N_{r+2}^mN_{r+4}^mN_{r+6}^m)^2}{\beta_{r+2}^m\beta_{r+4}^m\beta_{r+6}^m}+\dots$$

$$N_r^m=\frac{(2m+r)(2m+r-1)c^2}{(2m+2r-1)(2m+2r+1)}\frac{d_r}{d_{r-2}}\qquad (r\geq 2)$$

$$\beta_r^m=\frac{r(r-1)(2m+r)(2m+r-1)c^4}{(2m+2r-1)^2(2m+2r+1)(2m+2r-3)}\qquad (r\geq 2)$$