Wave Equation in Oblate Spheroidal Coordinates

21.5.2

$$\nabla^2\Phi+k^2\Phi=\frac{\partial}{\partial\xi}\left[(\xi^2+1)\frac{\partial\Phi}{\partial\xi}\right]+\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\Phi}{\partial\eta}\right]+\frac{\xi^2+\eta^2}{(\xi^2+1)(1-\eta^2)}\frac{\partial^2\Phi}{\partial\phi^2}+c^2(\xi^2+\eta^2)\Phi=0$$

$$\left(c=\frac{1}{2}fk\right)$$

21.5.2 may be obtained from **21.5.1** by the transformations

$$\xi\to\pm i\xi,\ c\to\mp ic.$$

21.6. Differential Equations for Radial and Angular Prolate Spheroidal Wave Functions

If in **21.5.1** we put

$$\Phi=R_{mn}(c,\xi)S_{mn}(c,\eta)\,{\cos \atop \sin}\,m\phi$$

then the "radial solution" $R_{mn}(c,\xi)$ and the "angular solution" $S_{mn}(c,\eta)$ satisfy the differential equations

21.6.1

$$\frac{d}{d\xi}\left[(\xi^2-1)\frac{d}{d\xi}R_{mn}(c,\xi)\right]-\left(\lambda_{mn}-c^2\xi^2+\frac{m^2}{\xi^2-1}\right)R_{mn}(c,\xi)=0$$

21.6.2

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}S_{mn}(c,\eta)\right]+\left(\lambda_{mn}-c^2\eta^2-\frac{m^2}{1-\eta^2}\right)S_{mn}(c,\eta)=0$$

where the separation constants (or eigenvalues) λ_{mn} are to be determined so that $R_{mn}(c,\xi)$ and $S_{mn}(c,\eta)$ are finite at $\xi=\pm1$ and $\eta=\pm1$ respectively.

(**21.6.1** and **21.6.2** are identical. Radial and angular prolate spheroidal functions satisfy the same differential equation over different ranges of the variable.)

Differential Equations for Radial and Angular Oblate Spheroidal Functions

21.6.3

$$\frac{d}{d\xi}\left[(\xi^2+1)\frac{d}{d\xi}R_{mn}(c,\xi)\right]-\left(\lambda_{mn}-c^2\xi^2-\frac{m^2}{\xi^2+1}\right)R_{mn}(c,\xi)=0$$

21.6.4

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}S_{mn}(c,\eta)\right]+\left(\lambda_{mn}+c^2\eta^2-\frac{m^2}{1-\eta^2}\right)S_{mn}(c,\eta)=0$$

(**21.6.3** may be obtained from **21.6.1** by the transformations $\xi\to\pm i\xi$, $c\to\mp ic$; **21.6.4** may be obtained from **21.6.2** by the transformation $c\to\mp ic$.)

21.7. Prolate Angular Functions

21.7.1

$$S^{(1)}_{mn}(c,\eta)=\sum_{r=0,1}^{\infty}{}' d^{mn}_r(c)P^m_{m+r}(\eta)$$

=Prolate angular function of the first kind

21.7.2

$$S^{(2)}_{mn}(c,\eta)=\sum_{r=-\infty}^{\infty}{}' d^{mn}_r(c)Q^m_{m+r}(\eta)$$

=Prolate angular function of the second kind

($P^m_n(\eta)$ and $Q^m_n(\eta)$ are associated Legendre functions of the first and second kinds respectively. However, for $-1\le z\le1$, $P^m_n(z)=(1-z^2)^{m/2}d^mP_n(z)/dz^m$ (see **8.6.6**). The summation is extended over even values or odd values of r.)

Recurrence Relations Between the Coefficients

21.7.3

$$\alpha_kd_{k+2}+(\beta_k-\lambda_{mn})d_k+\gamma_kd_{k-2}=0$$

$$\alpha_k=\frac{(2m+k+2)(2m+k+1)c^2}{(2m+2k+3)(2m+2k+5)}$$

$$\beta_k=(m+k)(m+k+1)+\frac{2(m+k)(m+k+1)-2m^2-1}{(2m+2k-1)(2m+2k+3)}c^2$$

$$\gamma_k=\frac{k(k-1)c^2}{(2m+2k-3)(2m+2k-1)}$$

Transcendental Equation for λ_{mn}

21.7.4

$$U(\lambda_{mn})=U_1(\lambda_{mn})+U_2(\lambda_{mn})=0$$

$$U_1(\lambda_{mn})=\gamma^m_r-\lambda_{mn}-\frac{\beta^m_r}{\gamma^m_{r-2}-\lambda_{mn}-}\,\frac{\beta^m_{r-2}}{\gamma^m_{r-4}-\lambda_{mn}-}\cdots$$

$$U_2(\lambda_{mn})=-\frac{\beta^m_{r+2}}{\gamma^m_{r+2}-\lambda_{mn}-}\,\frac{\beta^m_{r+4}}{\gamma^m_{r+4}-\lambda_{mn}-}\cdots$$

$$\beta^m_k=\frac{k(k-1)(2m+k)(2m+k-1)c^4}{(2m+2k-1)^2(2m+2k+1)(2m+2k-3)}\qquad(k\ge2)$$

$$\gamma^m_k=(m+k)(m+k+1)+\tfrac{1}{2}c^2\left[1-\frac{4m^2-1}{(2m+2k-1)(2m+2k+3)}\right]\qquad(k\ge0)$$

(The choice of r in **21.7.4** is arbitrary.)