

4.3.67

$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \dots$$

$$+ \frac{(-1)^{n-1} 2^{2n} (2^{2n}-1) B_{2n}}{(2n)!} z^{2n-1} + \dots \quad \left(|z| < \frac{\pi}{2}\right)$$

4.3.68

$$\csc z = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \frac{31z^5}{15120} + \dots$$

$$+ \frac{(-1)^{n-1} 2(2^{2n-1}-1) B_{2n}}{(2n)!} z^{2n-1} + \dots \quad (|z| < \pi)$$

4.3.69

$$\sec z = 1 + \frac{z^2}{2} + \frac{5z^4}{24} + \frac{61z^6}{720} + \dots$$

$$+ \frac{(-1)^n E_{2n}}{(2n)!} z^{2n} + \dots \quad \left(|z| < \frac{\pi}{2}\right)$$

4.3.70

$$\cot z = \frac{1}{z} - \frac{z}{3} + \frac{z^3}{45} - \frac{2z^5}{945} + \dots$$

$$- \frac{(-1)^{n-1} 2^{2n} B_{2n}}{(2n)!} z^{2n-1} + \dots \quad (|z| < \pi)$$

4.3.71

$$\ln \frac{\sin z}{z} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} B_{2n}}{n(2n)!} z^{2n} \quad (|z| < \pi)$$

4.3.72

$$\ln \cos z = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} (2^{2n}-1) B_{2n}}{n(2n)!} z^{2n} \quad (|z| < \frac{1}{2}\pi)$$

4.3.73

$$\ln \frac{\tan z}{z} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n-1}-1) B_{2n}}{n(2n)!} z^{2n}$$

$$(|z| < \frac{1}{2}\pi)$$

where B_n and E_n are the Bernoulli and Euler numbers (see chapter 23).

Limiting Values

4.3.74 $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

4.3.75 $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$

4.3.76 $\lim_{n \rightarrow \infty} n \sin \frac{x}{n} = x$

4.3.77 $\lim_{n \rightarrow \infty} n \tan \frac{x}{n} = x$

4.3.78 $\lim_{n \rightarrow \infty} \cos \frac{x}{n} = 1$

Inequalities

4.3.79 $\frac{\sin x}{x} > \frac{2}{\pi} \quad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$

4.3.80 $\sin x \leq x \leq \tan x \quad \left(0 \leq x \leq \frac{\pi}{2}\right)$

4.3.81 $\cos x \leq \frac{\sin x}{x} \leq 1 \quad (0 \leq x \leq \pi)$

4.3.82 $\pi < \frac{\sin \pi x}{x(1-x)} \leq 4 \quad (0 < x < 1)$

4.3.83 $|\sinh y| \leq |\sin z| \leq \cosh y$

4.3.84 $|\sinh y| \leq |\cos z| \leq \cosh y$

4.3.85 $|\csc z| \leq \operatorname{csch} |y|$

4.3.86 $|\cos z| \leq \cosh |z|$

4.3.87 $|\sin z| \leq \sinh |z|$

4.3.88 $|\cos z| < 2, \quad |\sin z| \leq \frac{6}{5}|z| \quad (|z| < 1)$

Infinite Products

4.3.89 $\sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2 \pi^2}\right)$

4.3.90 $\cos z = \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{(2k-1)^2 \pi^2}\right)$

Expansion in Partial Fractions

4.3.91 $\cot z = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{1}{z^2 - k^2 \pi^2}$
 $(z \neq 0, \pm \pi, \pm 2\pi, \dots)$

4.3.92 $\csc^2 z = \sum_{k=-\infty}^{\infty} \frac{1}{(z - k\pi)^2}$
 $(z \neq 0, \pm \pi, \pm 2\pi, \dots)$

4.3.93 $\csc z = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{(-1)^k}{z^2 - k^2 \pi^2}$
 $(z \neq 0, \pm \pi, \pm 2\pi, \dots)$

Continued Fractions

4.3.94 $\tan z = \frac{z}{1 - \frac{z^2}{3 - \frac{z^2}{5 - \frac{z^2}{7 - \dots}}}} \quad \left(z \neq \frac{\pi}{2} \pm n\pi\right)$

4.3.95

$$\tan az = \frac{a \tan z}{1 + \frac{(1-a^2) \tan^2 z}{3 + \frac{(4-a^2) \tan^2 z}{5 + \frac{(9-a^2) \tan^2 z}{7 + \dots}}}} \quad \left(-\frac{\pi}{2} < \mathcal{R} z < \frac{\pi}{2}, \quad az \neq \frac{\pi}{2} \pm n\pi\right)$$