Kernels $K_1(z, t)$ and $K_2(z, t)$

20.7.6 $K_1(z, t)=Z_\nu^{(j)}(u)[M(z, t)]^{-\nu/2}, \qquad (\mathscr{R}z>0)$

where

20.7.7 $u=\sqrt{2q(\cosh 2z+\cos 2t)}$

20.7.8 $M(z, t)=\cosh (z+it)/\cosh (z-it)$

To make $M^{-\frac{1}{2}\nu}$ single-valued, define

20.7.9

$$\cosh (z+i\pi)=e^{i\pi}\cosh z$$
$$\cosh (z-i\pi)=e^{-i\pi}\cosh z$$
$$M(z, 0)=1$$
$$[M(z, \pi)]^{-\frac{1}{2}\nu}=e^{-i\nu\pi}M(z, 0)$$

Let

20.7.10 $G(z,q)=\frac{1}{\pi}\int_0^\pi K_1(u,t)F_\nu(t)dt, \qquad (\mathscr{R}z>0)$

where $F_\nu(t)$ is defined in **20.3.8.** It may be verified that K_1F_ν satisfies **20.7.3,** K satisfies **20.7.2** and **20.7.4.** Hence G is a solution of **20.1.2** (with z replacing u). It can be shown that K_1 may be replaced by the more general function

20.7.11

$K_2(z,t)=Z^{(j)}_{\nu+2s}(u)[M(z,t)]^{-\frac{1}{2}\nu+s}$, s any integer.

See **20.4.7** for definition of $Z^{(j)}_{\nu+2s}(u)$.

From the known expansions for $Z^{(j)}_{\nu+2s}(u)$ when $\mathscr{R}z$ is large and positive it may be verified that

20.7.12

$$M_\nu^{(j)}(z, q)=\frac{(-1)^s}{\pi c_{2s}}\int_0^\pi Z^{(j)}_{\nu+2s}(u)\left[\frac{\cosh z+it}{\cosh z-it}\right]^{-\frac{1}{2}\nu-s}F_\nu(t)dt$$
$$(\mathscr{R}z>0, \mathscr{R}(\nu+\tfrac{1}{2})>0)$$

where $M_\nu^{(j)}(z, q)$ is given by **20.4.12,** $s=0, 1, \ldots$, $c_{2s}\neq 0$, and $F_\nu(t)$ is the Floquet solution, **20.3.8.**

Kernel $K_3(z, t, a)$

20.7.13 $K_3(z, t, a)=e^{2i\sqrt{q}\,w}$

where

20.7.14 $w=\cosh z\cos a\cos t+\sinh z\sin a\sin t$

20.7.15 $G(z, q, a)=\frac{1}{\pi}\oint_C e^{2i\sqrt{q}\,w}F_\nu(t)dt$

where $F_\nu(t)$ is the Floquet solution **20.3.8.** The path C is chosen so that $G(z, t, a)$ exists, and **20.7.2, 20.7.3** are satisfied. Then it may be verified that $K_3(z, t, a)$, considered as a function of z and t, satisfies **20.7.4;** also, considered as a function of a and t, K_3 satisfies **20.7.5.** Consequently $G(z, q, a)=Y(z, q)y(a, q)$, where Y and y satisfy **20.1.2** and **20.1.1,** respectively.

Choice of Path C. Three paths will be defined:

20.7.16

Path C_3: from $-d_1+i\infty$ to $d_2-i\infty$, d_1, d_2 real

$$-d_1<\arg[\sqrt{q}\{\cosh (z+ia)\pm 1\}]<\pi-d_1$$
$$-d_2<\arg[\sqrt{q}\{\cosh (z-ia)\pm 1\}]<\pi-d_2$$

20.7.17

Path C_4: from $d_2-i\infty$ to $2\pi+i\infty-d_1$

(same d_1, d_2 as in **20.7.16**)

20.7.18

$$F_\nu(a)M_\nu^{j}(z, q)=\frac{e^{-i\nu\frac{\pi}{2}}}{\pi}\oint_{C_j} e^{2i\sqrt{q}\,w}F_\nu(t)dt \qquad j=3, 4$$

where $M_\nu^{j}(z, q)$ is also given by **20.4.12.**

20.7.19 Path C_1: from $-d_1+i\infty$ to $2\pi-d_1+i\infty$

$$F_\nu(a)M_\nu^{(1)}(z, q)=\frac{e^{-i\nu\frac{\pi}{2}}}{2\pi}\oint_{C_1} e^{2i\sqrt{q}\,w}F_\nu(t)dt$$

See [20.36], section **2.68.**

If ν is an integer the paths can be simplified; for in that case $F_\nu(t)$ is periodic and the integrals exist when the path is taken from 0 to 2π. Still further simplifications are possible, if z is also real.

The following are among the more important integral representations for the periodic functions $ce_r(z, q)$, $se_r(z, q)$ and for the associated radial solutions.

Let $r=2s+p$, $p=0$ or 1

20.7.20

$$ce_r(z,q)=\rho_r\int_0^{\pi/2}\cos\left(2\sqrt{q}\cos z\cos t-p\frac{\pi}{2}\right)ce_r(t, q)dt$$