20.6.2 $$Se_{2r+p}(z,q)=-ise_{2r+p}(iz,q)=\sum_{k=0}^{\infty} B_{2k+p}^{2r+p}(q)\sinh(2k+p)z,\text{ associated with } b_r$$

writing $A_{2k+p}^{2r+p}(q)=A_{2k+p}$ for brevity; similarly for B_{2k+p}; $p=0, 1,$

20.6.3 $$Ce_{2r}(z,q)=\frac{ce_{2r}\left(\frac{\pi}{2},q\right)}{A_0^{2r}}\sum_{k=0}^{\infty}(-1)^k A_{2k}J_{2k}(2\sqrt{q}\cosh z)=\frac{ce_{2r}(0,q)}{A_0^{2r}}\sum_{k=0}^{\infty}A_{2k}J_{2k}(2\sqrt{q}\sinh z)$$

20.6.4 $$Ce_{2r+1}(z,q)=\frac{ce'_{2r+1}\left(\frac{\pi}{2},q\right)}{\sqrt{q}A_1^{2r+1}}\sum_{k=0}^{\infty}(-1)^{k+1}A_{2k+1}J_{2k+1}(2\sqrt{q}\cosh z)$$

$$=\frac{ce_{2r+1}(0,q)}{\sqrt{q}A_1^{2r+1}}\coth z\sum_{k=0}^{\infty}(2k+1)A_{2k+1}J_{2k+1}(2\sqrt{q}\sinh z)$$

20.6.5 $$Se_{2r}(z,q)=\frac{se'_{2r}\left(\frac{\pi}{2},q\right)\tanh z}{qB_2^{2r}}\sum_{k=1}^{\infty}(-1)^k 2kB_{2k}J_{2k}(2\sqrt{q}\cosh z)$$

$$=\frac{se'_{2r}(0,q)}{qB_2^{2r}}\coth z\sum_{k=1}^{\infty}2kB_{2k}J_{2k}(2\sqrt{q}\sinh z)$$

20.6.6 $$Se_{2r+1}(z,q)=\frac{se_{2r+1}\left(\frac{\pi}{2},q\right)}{\sqrt{q}B_1^{2r+1}}\tanh z\sum_{k=0}^{\infty}(-1)^k(2k+1)B_{2k+1}J_{2k+1}(2\sqrt{q}\cosh z)$$

$$=\frac{se'_{2r+1}(0,q)}{\sqrt{q}B_1^{2r+1}}\sum_{k=0}^{\infty}B_{2k+1}J_{2k+1}(2\sqrt{q}\sinh z)$$

See [20.30] for still other forms.

Solutions of the second kind, as well as solutions of the third and fourth kind (analogous to Hankel functions) are obtainable from **20.4.12.**

20.6.7 $$Mc_{2r}^{(j)}(z,q)=\sum_{k=0}^{\infty}(-1)^{r+k}A_{2k}^{2r}(q)[J_{k-s}(u_1)Z_{k+s}^{(j)}(u_2)+J_{k+s}(u_1)Z_{k-s}^{(j)}(u_2)]/\epsilon_s A_{2s}^{2r}$$

where $\epsilon_0=2$, $\epsilon_s=1$, for $s=1, 2, \ldots$; s arbitrary, associated with a_{2r}

20.6.8 $$Mc_{2r+1}^{(j)}(z,q)=\sum_{k=0}^{\infty}(-1)^{r+k}A_{2k+1}^{2r+1}(q)[J_{k-s}(u_1)Z_{k+s+1}^{(j)}(u_2)+J_{k+s+1}(u_1)Z_{k-s}^{(j)}(u_2)]/A_{2s+1}^{2r+1}$$

associated with a_{2r+1}

20.6.9 $$Ms_{2r}^{(j)}(z,q)=\sum_{k=1}^{\infty}(-1)^{k+r}B_{2k}^{2r}(q)[J_{k-s}(u_1)Z_{k+s}^{(j)}(u_2)-J_{k+s}(u_1)Z_{k-s}^{(j)}(u_2)]/B_{2s}^{2r},\text{ associated with } b_{2r}$$

20.6.10 $$Ms_{2r+1}^{(j)}(z,q)=\sum_{k=0}^{\infty}(-1)^{k+r}B_{2k+1}^{2r+1}(q)[J_{k-s}(u_1)Z_{k+s+1}^{(j)}(u_2)-J_{k+s+1}(u_1)Z_{k-s}^{(j)}(u_2)]/B_{2s+1}^{2r+1}$$

associated with b_{2r+1}

where

$$u_1=\sqrt{q}e^{-z},\quad u_2=\sqrt{q}e^{z},\quad B_{2s+p}^{2r+p},\ A_{2s+p}^{2r+p}\neq 0,\ p=0,1.$$

See **20.4.7** for definition of $Z_m^{(j)}(x)$.

Solutions **20.6.7–20.6.10** converge for all values of z, when $q\neq 0$. If $j=2, 3, 4$ the logarithmic terms entering into the Bessel functions $Y_m(u_2)$ must be defined, to make the functions single-valued. This can be accomplished as follows:

Define (as in [20.58])

20.6.11 $$\ln(\sqrt{q}e^{z})=\ln(\sqrt{q})+z$$

See [20.15] and [20.36], section **2.75** for derivation.