where the coefficients c_{2n} are those associated with Floquet's solution. In the above, ν may be complex. Except for the special case when ν is an integer, the following holds:

$$\frac{\varphi_{2n+\nu-2}}{\varphi_{2n+\nu}} \sim \frac{\varphi_{-2n+\nu}}{\varphi_{-2n+\nu+2}} \sim \frac{-4n^2}{q[\cos(z-b)]^2} \qquad (n\to\infty)$$

If ν and n are integers, $J_{-2n+\nu}(f)=(-1)^\nu J_{2n-\nu}(f)$.

$$[\varphi_{2n+\nu}/\varphi_{2n+\nu-2}] \sim -[\cos(z-b)]^2 q/4n^2$$

$$[\varphi_{-2n+\nu}/\varphi_{-2n+\nu+2}] \sim -4n^2/q\,[\cos(z-b)]^2$$

On the other hand

$$\frac{c_{2n}}{c_{2n-2}} \sim \frac{c_{-2n}}{c_{-2n+2}} \sim \frac{-q}{4n^2} \qquad (n\to\infty)$$

It follows that **20.4.4** converges absolutely and uniformly in every closed region where

$|\cos(z-b)|>d_1>1$.

There are two such disjoint regions:

(I) $\mathscr{I}(z-b)>d_2>0;\quad (|\cos(z-b)|>d_1>1)$

(II) $\mathscr{I}(z-b)<-d_2<0;\quad (|\cos(z-b)|>d_1>1)$

If ν is an integer **20.4.4** converges for all values of z. Various representations are found by specializing b.

20.4.5

$$\text{If } b=0,\ y=e^{i\pi\nu/2}\sum_{n=-\infty}^{\infty} c_{2n}(-1)^n J_{2n+\nu}(2\sqrt{q}\cos z) \qquad (|\cos z|>1,\ |\arg 2\sqrt{q}\cos z|\le\pi)$$

20.4.6

$$\text{If } b=\frac{\pi}{2},\ y=\sum_{n=-\infty}^{\infty} c_{2n}J_{2n+\nu}(2i\sqrt{q}\sin z) \qquad (|\sin z|>1,\ |\arg 2\sqrt{q}\sin z|\le\pi)$$

If $b\to\infty i$, y reduces to a multiple of the solution **20.3.8.** The fact that **20.3.8, 20.4.5,** and **20.4.6** are special cases of **20.4.4** explains why it is that these apparently dissimilar expansions involve the same set of coefficients c_{2n}.

Since **20.4.4** results from the recurrence properties of Bessel functions, $J_k(f)$ can be replaced by $H_k^{(j)}(f)$, $j=1, 2$, where $H_k^{(j)}$ is the Hankel function, at least formally. Thus let

$$\psi_k^j=[e^{i\pi}\cos(z-b)/\cos(z+b)]^{\frac{1}{2}k}H_k^{(j)}(f)$$

where f satisfies **20.4.2.** An examination of the ratios $\psi_{2n+\nu}/\psi_{2n+\nu-2}$ shows that

$$y=\sum_{n=-\infty}^{\infty} c_{2n}\psi_{2n+\nu}^{(j)}$$

will be a solution provided

$$|\cos(z-b)|>1;\ |\cos(z+b)|>1.$$

The above two conditions are necessary even when ν is an integer. Once b is fixed, the regions in which the solutions converge can be readily established.

Following [20.36] let

20.4.7

$$J_p(x)=Z_p^{(1)}(x);\quad Y_p(x)=Z_p^{(2)}(x);\quad H_p^{(1)}(x)=Z_p^{(3)}(x);\quad H_p^{(2)}(x)=Z_p^{(4)}(x)$$

If z is replaced by $-iz$ in **20.4.5** and **20.4.6** solutions of **20.1.2** are obtained. Thus

20.4.8

$$y_1^{(j)}(z)=\sum_{n=-\infty}^{\infty} c_{2n}(-1)^n Z_{2n+\nu}^{(j)}(2\sqrt{q}\cosh z) \qquad (|\cosh z|>1)$$

20.4.9

$$y_2^{(j)}(z)=\sum_{n=-\infty}^{\infty} c_{2n}Z_{2n+\nu}^{(j)}(2\sqrt{q}\sinh z) \qquad (|\sinh z|>1,\ j=1, 2, 3, 4)$$

The relation between $y_1^{(j)}(z)$ and $y_2^{(j)}(z)$ can be determined from the asymptotic properties of the Bessel functions for large values of argument. It can be shown that

20.4.10

$$y_1^{(j)}(z)/y_2^{(j)}(z)=[F_\nu(0)/F_\nu\left(\frac{\pi}{2}\right)]e^{i\nu\pi/2} \qquad (\mathscr{R}z>0):$$

When ν is not an integer, the above solutions do not vanish identically. See **20.6** for integral values of ν.

Solutions Involving Products of Bessel Functions

20.4.11

$$y_3^{(j)}(z)=\frac{1}{c_{2s}}\sum_{n=-\infty}^{\infty} c_{2n}(-1)^n Z_{n+\nu+s}^{(j)}(\sqrt{q}e^{iz})J_{n-s}(\sqrt{q}e^{-iz}) \qquad (j=1, 2, 3, 4)$$

satisfies **20.1.1,** where $Z_n^{(j)}(u)$ is defined in **20.4.7,** the coefficients c_{2n} belong to the Floquet solution, and s is an arbitrary integer, $c_{2s}\neq 0$. The solution converges over the entire complex z-plane if $q\neq 0$. Written with z replaced by $-iz$, one obtains solutions of **20.1.2.**