systematic tabulation, this method is considerably faster than the method of numerical integration. Thus, when **20.3.8** is substituted into **20.3.1**, there result the following recurrence relations:

20.3.12 $$V_{2n}c_{2n}=c_{2n-2}+c_{2n+2}$$

where

20.3.13 $V_{2n}=[a-(2n+\nu)^2]/q, \quad -\infty<n<\infty.$

When ν is complex, the coefficients V_{2n} may also be complex. As in **20.2**, it is possible to generate the ratios

$$G_m=c_m/c_{m-2} \text{ and } H_{-m}=c_{-m-2}/c_{-m}$$

from the continued fractions

20.3.14

$$G_m=\frac{1}{V_m-}\,\frac{1}{V_{m+2}-\ \ldots}, \qquad m\geq 0$$

$$H_{-m}=\frac{1}{V_{-m-2}-}\,\frac{1}{V_{-m-4}-\ \ldots}, \qquad m\geq 0.$$

From the form of **20.3.13** and the known properties of continued fractions it is assured that for sufficiently large values of $|m|$ both $|G_m|$ and $|H_{-m}|$ converge. Once values of G_m and H_{-m} are available for some sufficiently large value of m, then the finite number of ratios $G_{m-2}, G_{m-4}, \ldots, G_0$ can be computed in turn, if they exist. Similarly for $H_{-m+2}, \ldots, H_0$. It is easy to show that ν is the correct characteristic exponent, appropriate for the point (a, q), if and only if $H_0G_0=1$. An iteration technique can be used to improve the value of ν, by the method suggested in [20.3]. One coefficient c_j can be assigned arbitrarily; the rest are then completely determined. After all the c_j become available, a multiplier (depending on q but not on z) can be found to satisfy a prescribed normalization.

It is well known that continued fractions can be converted to determinantal form. Equation **20.3.14** can in fact be written as a determinant with an infinite number of rows—a special case of Hill's determinant. See [20.19], [20.36], [20.15], or [20.30] for details. Although the determinant has actually been used in computations where high-speed computers were available, the direct use of the continued fraction seems much less laborious.

Special Cases (*a, q* Real)

Corresponding to $q=0$, $y_1=\cos\sqrt{a}z$, $y_2=\sin\sqrt{a}z$; the Floquet solutions are $\exp(iaz)$ and $\exp(-iaz)$. As a, q vary continuously in the $q-a$ plane, ν describes curves; ν is real when (q, a), $q\geq 0$ lies in the region between $a_r(q)$ and $b_{r+1}(q)$ and

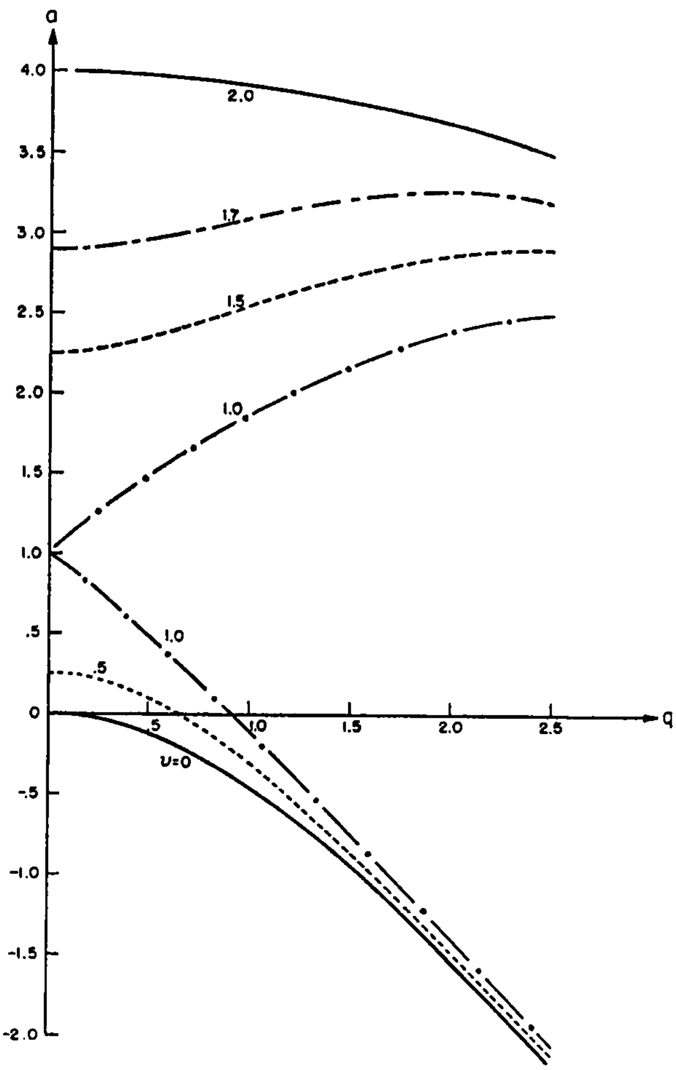

FIGURE 20.6. *Characteristic Exponent-First Two Stable Regions $y=e^{i\nu x}P(x)$ where $P(x)$ is a periodic function of period π.*

Definition of ν;
In first stable region, $0\leq\nu\leq 1$,
In second stable region, $1\leq\nu\leq 2$.

(Constructed from tabular values supplied by T. Tamir, Brooklyn Polytechnic Institute)

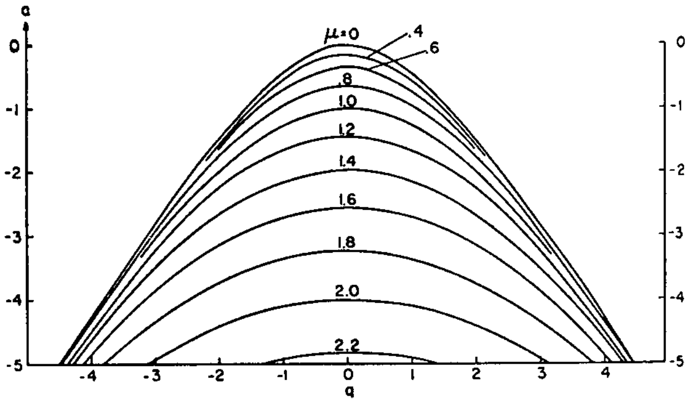

FIGURE 20.7. *Characteristic Exponent in First Unstable Region. Differential equation: $y''+(a-2q\cos 2x)y=0$. The Floquet solution $y=e^{i\nu x}P(x)$, where $P(x)$ is a periodic function of period π. In the first unstable region, $\nu=i\mu$; μ is given for $a\geq -5$.* (Constructed at NBS.)