The above expansion is not limited to integral values of r, and it is a very good approximation for r of the form $n+\frac{1}{2}$ where n is an integer. In case of integral values of $r=n$, the series holds only up to terms not involving r^2-n^2 in the denominator. Subsequent terms must be derived specially (as shown by Mathieu). Mulholland and Goldstein [20.38] have computed characteristic values for purely imaginary q and found that a_0 and a_2 have a common real value for $|q|$ in the neighborhood of 1.468; Bouwkamp [20.5] has computed this number as $q_0=\pm i\ 1.46876852$ to 8 decimals. For values of $-iq>-iq_0$, a_0 and a_2 are conjugate complex numbers. From equation **20.2.25** it follows that the radius of convergence for the series defining a_0 is no greater than $|q_0|$. It is shown in [20.36], section **2.25** that the radius of convergence for $a_{2n}(q)$, $n\geq 2$ is greater than 3. Furthermore

$$a_r-b_r=O(q^r/r^{r-1}),\ r\rightarrow\infty.$$

Power Series in q for the Periodic Functions (for sufficiently small $|q|$)

20.2.27

$$ce_0(z,q)=2^{-\frac{1}{2}}\left[1-\frac{q}{2}\cos 2z+q^2\left(\frac{\cos 4z}{32}-\frac{1}{16}\right)-q^3\left(\frac{\cos 6z}{1152}-\frac{11\cos 2z}{128}\right)+\dots\right]$$

$$ce_1(z,q)=\cos z-\frac{q}{8}\cos 3z+q^2\left[\frac{\cos 5z}{192}-\frac{\cos 3z}{64}-\frac{\cos z}{128}\right]-q^3\left[\frac{\cos 7z}{9216}-\frac{\cos 5z}{1152}-\frac{\cos 3z}{3072}+\frac{\cos z}{512}\right]+\dots$$

$$se_1(z,q)=\sin z-\frac{q}{8}\sin 3z+q^2\left[\frac{\sin 5z}{192}+\frac{\sin 3z}{64}-\frac{\sin z}{128}\right]-q^3\left[\frac{\sin 7z}{9216}+\frac{\sin 5z}{1152}-\frac{\sin 3z}{3072}-\frac{\sin z}{512}\right]+\dots$$

$$ce_2(z,q)=\cos 2z-q\left(\frac{\cos 4z}{12}-\frac{1}{4}\right)+q^2\left(\frac{\cos 6z}{384}-\frac{19\cos 2z}{288}\right)+\dots$$

$$se_2(z,q)=\sin 2z-q\frac{\sin 4z}{12}+q^2\left(\frac{\sin 6z}{384}-\frac{\sin 2z}{288}\right)+\dots$$

20.2.28

$$\begin{matrix}ce_r(z,q)\\ se_r(z,q)\end{matrix}=\cos(rz-p(\pi/2))-q\left\{\frac{\cos\left[(r+2)z-p\frac{\pi}{2}\right]}{4(r+1)}-\frac{\cos[(r-2)z-p(\pi/2)]}{4(r-1)}\right\}$$

$$+q^2\left\{\frac{\cos[(r+4)z-p(\pi/2)]}{32(r+1)(r+2)}+\frac{\cos[(r-4)z-p(\pi/2)]}{32(r-1)(r-2)}-\frac{\cos[rz-p(\pi/2)]}{32}\left[\frac{2(r^2+1)}{(r^2-1)^2}\right]\right\}+\dots$$

with $p=0$ for $ce_r(z,q)$, $p=1$ for $se_r(z,q)$, $r\geq 3$.

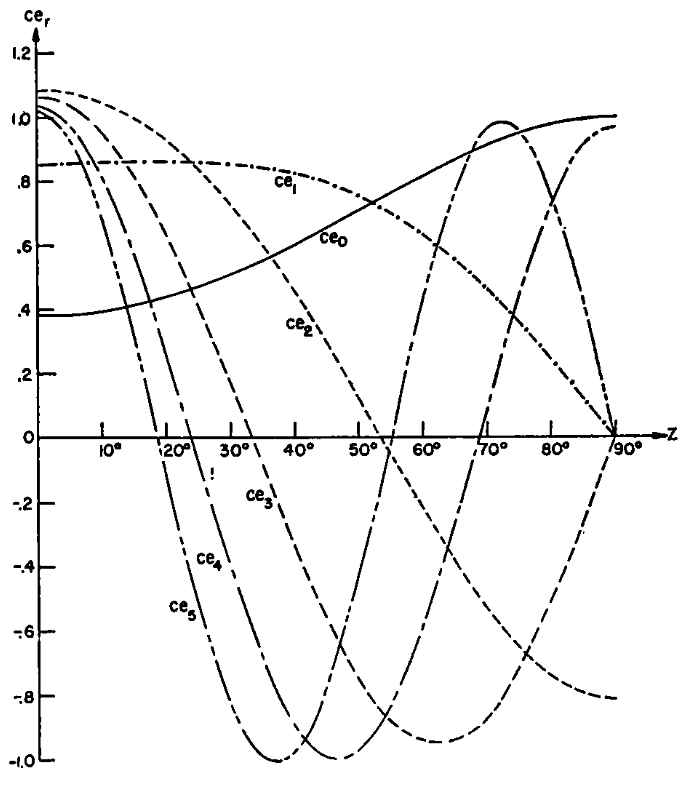

FIGURE 20.2. *Even Periodic Mathieu Functions, Orders 0–5* $q=1$.