19.21.7 $u_r+iv_r=\Gamma(r+\tfrac{1}{2}+ia)/\Gamma(\tfrac{1}{2}+ia)$

or

19.21.8 $s(a, x)\sim\sum_{r=0}^{\infty} (-i)^r \frac{\Gamma(2r+\frac{1}{2}+ia)}{\Gamma(\frac{1}{2}+ia)} \frac{1}{2^r r! x^{2r}}$

19.22. Expansions for a Large With x Moderate

(i) a positive

When $a>>x^2$, with $p=\sqrt{a}$, then

19.22.1 $W(a, x)=W(a, 0) \exp(-px+v_1)$

19.22.2 $W(a,-x)=W(a, 0) \exp(px+v_2)$

where $W(a, 0)$ is given by **19.17.4,** and

19.22.3

$$v_1, v_2\sim\pm\frac{\frac{2}{3}(\frac{1}{2}x)^3}{2p}+\frac{(\frac{1}{2}x)^2}{(2p)^2}\pm\frac{\frac{1}{2}x+\frac{2}{5}(\frac{1}{2}x)^5}{(2p)^3}+\frac{2(\frac{1}{2}x)^4}{(2p)^4}\pm\frac{\frac{16}{3}(\frac{1}{2}x)^3+\frac{4}{7}(\frac{1}{2}x)^7}{(2p)^5}+\ldots \qquad (a\to+\infty)$$

The upper sign gives the first function, and the lower sign the second function.

(ii) a negative

When $-a>>x^2$, with $p=\sqrt{-a}$, then

19.22.4

$$W(a, x)+iW(a, -x)=\sqrt{2}W(a, 0) \exp\{v_r+i(px+\tfrac{1}{4}\pi+v_i)\}$$

where $W(a, 0)$ is given by **19.17.4,** and

19.22.5

$$v_r\sim-\frac{(\frac{1}{2}x)^2}{(2p)^2}+\frac{2(\frac{1}{2}x)^4}{(2p)^4}-\frac{9(\frac{1}{2}x)^2+\frac{16}{3}(\frac{1}{2}x)^6}{(2p)^6}+\ldots$$

$$v_i\sim\frac{\frac{2}{3}(\frac{1}{2}x)^3}{2p}-\frac{\frac{1}{2}x+\frac{2}{5}(\frac{1}{2}x)^5}{(2p)^3}+\frac{\frac{16}{3}(\frac{1}{2}x)^3+\frac{4}{7}(\frac{1}{2}x)^7}{(2p)^5}-\ldots \qquad (a\to-\infty)$$

Further expansions of a similar type will be found in [19.3].

19.23. Darwin's Expansions

(i) a positive, $x^2-4a>>0$

Write

19.23.1

$$X=\sqrt{x^2-4a} \qquad \theta=4a\vartheta_2(x/2\sqrt{a})=\tfrac{1}{2}\int_{2\sqrt{a}}^{x} Xdx$$
$$=\tfrac{1}{4}xX-a\ln\frac{x+X}{2\sqrt{a}}$$
$$=\tfrac{1}{4}x\sqrt{x^2-4a}-a \operatorname{arccosh}\frac{x}{2\sqrt{a}}$$

(see **Table 19.3** for ϑ_2), then

19.23.2 $W(a, x)=\sqrt{2k}e^{v_r}\cos(\tfrac{1}{4}\pi+\theta+v_i)$

19.23.3 $W(a, -x)=\sqrt{2/k}e^{v_r}\sin(\tfrac{1}{4}\pi+\theta+v_i)$

where

19.23.4

$$v_r\sim-\tfrac{1}{2}\ln X-\frac{d_6}{X^6}+\frac{d_{12}}{X^{12}}-\ldots$$
$$v_i\sim-\frac{d_3}{X^3}+\frac{d_9}{X^9}-\frac{d_{15}}{X^{15}}+\ldots \qquad (x^2-4a\to\infty)$$

and d_{3r} is given by **19.23.12.**

(ii) a positive, $4a-x^2>>0$

Write

19.23.5

$$Y=\sqrt{4a-x^2} \qquad \theta=4a\vartheta_4(x/2\sqrt{a})$$
$$=\tfrac{1}{2}\int_0^x Ydx=\tfrac{1}{4}xY+a\arcsin\frac{x}{2\sqrt{a}}$$

(see **Table 19.3** for $\vartheta_4=\frac{1}{8}\pi-\vartheta_3$), then

19.23.6 $W(a, x)=\exp\{-\theta+v(a, x)\}$

19.23.7 $W(a, -x)=\exp\{\theta+v(a, -x)\}$

where

19.23.8

$$v(a, x)\sim-\tfrac{1}{2}\ln Y+\frac{d_3}{Y^3}+\frac{d_6}{Y^6}+\frac{d_9}{Y^9}+\ldots \qquad (x^2-4a\to-\infty)$$

and d_{3r} is again given by **19.23.12.**

(iii) a negative, $x^2-4a>>0$

Write

19.23.9

$$X=\sqrt{x^2+4|a|} \qquad \theta=4|a|\vartheta_1(x/2\sqrt{|a|})=\tfrac{1}{2}\int_0^x Xdx$$
$$=\tfrac{1}{4}xX-a\ln\frac{x+X}{2\sqrt{|a|}}$$
$$=\tfrac{1}{4}x\sqrt{x^2+4|a|}-a \operatorname{arcsinh}\frac{x}{2\sqrt{|a|}}$$

(see **Table 19.3** for ϑ_1) then

19.23.10 $W(a, x)=\sqrt{2k}e^{v_r}\cos(\tfrac{1}{4}\pi+\theta+v_i)$

19.23.11 $W(a, -x)=\sqrt{2/k}e^{v_r}\sin(\tfrac{1}{4}\pi+\theta+v_i)$

where v_r and v_i are again given by **19.23.4.** In each case the coefficients d_{3r} are given by