(see **Table 19.3** for ϑ_1), then

19.10.2 $$U(a,x)=\frac{(2\pi)^{1/4}}{\sqrt{\Gamma(\frac{1}{2}+a)}}\exp\{-\theta+v(a,x)\}$$

19.10.3 $$U(a,-x)=\frac{(2\pi)^{1/4}}{\sqrt{\Gamma(\frac{1}{2}+a)}}\exp\{\theta+v(a,-x)\}$$

where

19.10.4

$$v(a,x)\sim-\tfrac{1}{2}\ln X+\sum_{s=1}(-1)^s d_{3s}/X^{3s} \qquad (a>0,\ x^2+4a\to+\infty)$$

and d_{3s} is given by **19.10.13.**

(ii) a negative, x^2+4a large and positive. Write

19.10.5 $$X=\sqrt{x^2-4|a|}$$

$$\theta=4|a|\vartheta_2(x/2\sqrt{|a|})=\tfrac{1}{2}\int_{2\sqrt{|a|}}^{x}Xdx=\tfrac{1}{4}xX+a\ln\frac{x+X}{2\sqrt{|a|}}$$

$$=\tfrac{1}{4}x\sqrt{x^2-4|a|}+a\operatorname{arccosh}\frac{x}{2\sqrt{|a|}}$$

(see **Table 19.3** for ϑ_2), then

19.10.6 $$U(a,x)=\frac{\sqrt{\Gamma(\frac{1}{2}-a)}}{(2\pi)^{1/4}}\exp\{-\theta+v(a,x)\}$$

19.10.7

$$V(a,x)=\frac{2}{(2\pi)^{1/4}\sqrt{\Gamma(\frac{1}{2}-a)}}\exp\{\theta+v(a,-x)\}$$

where again

19.10.8

$$v(a,x)\sim-\tfrac{1}{2}\ln X+\sum_{s=1}(-1)^s d_{3s}/X^{3s} \qquad (a<0,\ x^2+4a\to+\infty)$$

and d_{3s} is given by **19.10.13.**

(iii) a large and negative and x moderate. Write

19.10.9 $$Y=\sqrt{4|a|-x^2}$$

$$\theta=4|a|\vartheta_4(x/2\sqrt{|a|})$$

$$=\tfrac{1}{2}\int_0^x Ydx=\tfrac{1}{4}xY+|a|\arcsin\frac{x}{2\sqrt{|a|}}$$

(see **Table 19.3** for $\vartheta_4=\frac{1}{8}\pi-\vartheta_3$), then

19.10.10

$$U(a,x)=\frac{2\sqrt{\Gamma(\frac{1}{2}-a)}}{(2\pi)^{1/4}}e^{v_r}\cos\left\{\tfrac{1}{4}\pi+\tfrac{1}{2}\pi a+\theta+v_i\right\}$$

19.10.11

$$V(a,x)=\frac{2}{(2\pi)^{\frac{1}{4}}\sqrt{\Gamma(\frac{1}{2}-a)}}e^{v_r}\sin\{\tfrac{1}{4}\pi+\tfrac{1}{2}\pi a+\theta+v_i\}$$

where

19.10.12 $$v_r\sim-\tfrac{1}{2}\ln Y-\frac{d_6}{Y^6}+\frac{d_{12}}{Y^{12}}-\ldots$$

$$v_i\sim\frac{d_3}{Y^3}-\frac{d_9}{Y^9}+\ldots \qquad (x^2+4a\to-\infty)$$

In each case the coefficients d_{3r} are given by

19.10.13

$$d_3=\frac{1}{a}\left(\frac{x^3}{48}+\tfrac{1}{2}ax\right)$$

$$d_6=\tfrac{3}{4}x^2-2a$$

$$d_9=\frac{1}{a^3}\left(-\frac{7}{5760}x^9-\frac{7}{320}ax^7-\frac{49}{320}a^2x^5+\frac{31}{12}a^3x^3-19a^4x\right)$$

$$d_{12}=\frac{153}{8}x^4-186ax^2+80a^2$$

See [19.11] for $d_{15},\ldots,d_{24}$, and [19.5] for an alternative form.

19.11. Modulus and Phase

When a is negative and $|x|<2\sqrt{|a|}$, the functions U and V are oscillatory and it is sometimes convenient to write

19.11.1 $$U(a,x)+i\Gamma(\tfrac{1}{2}-a)V(a,x)=F(a,x)e^{i\chi(a,x)}$$

19.11.2 $$U'(a,x)+i\Gamma(\tfrac{1}{2}-a)V'(a,x)=-G(a,x)e^{i\psi(a,x)}$$

Then, when $a<0$ and $|a|>>x^2$,

19.11.3

$$F=\frac{\Gamma(\frac{1}{4}-\frac{1}{2}a)}{2^{\frac{1}{2}a+\frac{1}{4}}\sqrt{\pi}}e^{v_r}, \qquad \chi=(\tfrac{1}{2}a+\tfrac{1}{4})\pi+px+v_i$$

where v_r, v_i are given by **19.9.5** and $p=\sqrt{-a}$.

Alternatively, with $p=\sqrt{|a|}$, and again $-a>>x^2$,

19.11.4

$$F\sim\frac{\Gamma(\frac{1}{4}-\frac{1}{2}a)}{2^{\frac{1}{2}a+\frac{1}{4}}\sqrt{\pi}}\left\{1+\frac{x^2}{(4p)^2}+\frac{\frac{5}{2}x^4}{(4p)^4}+\frac{\frac{15}{2}x^6-144x^2}{(4p)^6}+\ldots\right\}$$