19.5.4 $U(a,z)=\frac{1}{\sqrt{2\pi i}}\,e^{\frac14 z^2}\int_{\epsilon} e^{-zs+\frac12 s^2}s^{-a-\frac12}ds$

19.5.5 $=\frac{e^{(a-\frac12)\pi i}}{\sqrt{2\pi i}}\,e^{\frac14 z^2}\int_{\epsilon_3} e^{zs+\frac12 s^2}s^{-a-\frac12}ds$

19.5.6 $=\frac{e^{-(a-\frac12)\pi i}}{\sqrt{2\pi i}}\,e^{\frac14 z^2}\int_{\epsilon_4} e^{zs+\frac12 s^2}s^{-a-\frac12}ds$

where ϵ, ϵ_3 and ϵ_4 are shown in **Figures 19.3** and **19.4.**

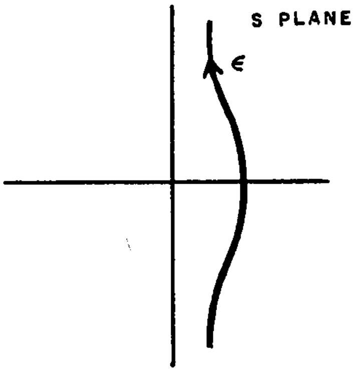

FIGURE 19.3

$-\frac12\pi<\arg s<\frac12\pi$

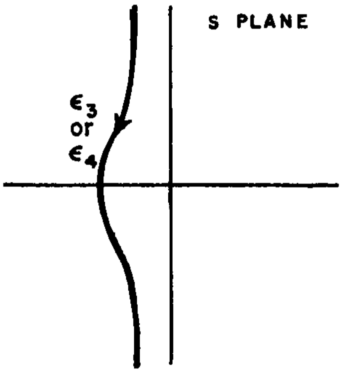

FIGURE 19.4

On ϵ_3 $\frac12\pi<\arg s<\frac32\pi$

On ϵ_4 $-\frac32\pi<\arg s<-\frac12\pi$

19.5.7

$$U(a,z)=\frac{\Gamma(\frac34-\frac12 a)}{2^{\frac12 a+\frac14}\pi i}\int_{(\zeta_1)} e^{\frac14 z^2 t}(1+t)^{\frac12 a-\frac34}(1-t)^{-\frac12 a-\frac34}dt$$

19.5.8

$$=\frac{\Gamma(\frac34-\frac12 a)}{2^{\frac12 a+\frac14}\pi i}\int_{\zeta_1}\tfrac12 z e^{v}(\tfrac14 z^2+v)^{\frac12 a-\frac34}(\tfrac14 z^2-v)^{-\frac12 a-\frac34}dv$$

19.5.9

$U(a,z)$

$$=\frac{i\Gamma(\frac14-\frac12 a)}{2^{\frac12 a+\frac34}\pi}\int_{(\eta_1)}\tfrac12 z e^{-\frac14 z^2 t}(1+t)^{-\frac12 a-\frac14}(1-t)^{\frac12 a-\frac14}dt$$

19.5.10

$$=\frac{i\Gamma(\frac14-\frac12 a)}{2^{\frac12 a+\frac34}\pi}\int_{\eta_1} e^{-v}(\tfrac14 z^2+v)^{-\frac12 a-\frac14}(\tfrac14 z^2-v)^{\frac12 a-\frac14}dv$$

The contour ζ_1 is such that $(\frac14 z^2+v)$ goes from $\infty e^{-i\pi}$ to $\infty e^{i\pi}$ while $v=\frac14 z^2$ is *not* encircled; $(\frac14 z^2-v)^{-\frac12 a-\frac34}$ has its principal value except possibly in the immediate neighborhood of the branch-point when encirclement is being avoided. Likewise η_1 is such that $(\frac14 z^2-v)$ goes from $\infty e^{i\pi}$ to $\infty e^{-i\pi}$ while encirclement of $v=-\frac14 z^2$ is similarly avoided. The contours (ζ_1) and (η_1) may be obtained from ζ_1 and η_1 by use of the substitution $v=\frac14 z^2 t$.

The expressions **19.5.7** and **19.5.8** become indeterminate when $a=\frac32, \frac72, \frac{11}{2}, \ldots$; for these values

19.5.11

$$U(a,z)=\frac{1}{\Gamma(\frac34+\frac12 a)}\,z e^{-\frac14 z^2}\int_0^\infty e^{-s}s^{\frac12 a-\frac14}(z^2+2s)^{-\frac12 a-\frac34}ds$$

Again **19.5.9** and **19.5.10** become indeterminate when $a=\frac12, \frac52, \frac92, \ldots$; for these values

19.5.12

$$U(a,z)=\frac{1}{\Gamma(\frac14+\frac12 a)}\,e^{-\frac14 z^2}\int_0^\infty e^{-s}s^{\frac12 a-\frac34}(z^2+2s)^{-\frac12 a-\frac14}ds$$

Barnes-Type Integrals

19.5.13 $U(a,z)=\frac{e^{-\frac14 z^2}}{2\pi i}\,z^{-a-\frac12}\int_{-\infty i}^{+\infty i}\frac{\Gamma(s)\Gamma(\frac12+a-2s)}{\Gamma(\frac12+a)}(\sqrt2 z)^{2s}ds$ $\qquad(|\arg z|<\frac34\pi)$

where the contour separates the zeros of $\Gamma(s)$ from those of $\Gamma(a+\frac12-2s)$. Similarly

19.5.14 $V(a,z)=\sqrt{\frac{2}{\pi}}\,\frac{e^{\frac14 z^2}}{2\pi i}\,z^{a-\frac12}\int_{-\infty i}^{+\infty i}\frac{\Gamma(s)\Gamma(\frac12-a-2s)}{\Gamma(\frac12-a)}(\sqrt2 z)^{2s}\cos s\pi\, ds$ $\qquad(|\arg z|<\frac14\pi)$

19.6. Recurrence Relations

19.6.1 $U'(a,x)+\frac12 xU(a,x)+(a+\frac12)U(a+1,x)=0$

19.6.2 $U'(a,x)-\frac12 xU(a,x)+U(a-1,x)=0$

19.6.3 $2U'(a,x)+U(a-1,x)+(a+\frac12)U(a+1,x)=0$

19.6.4 $xU(a,x)-U(a-1,x)+(a+\frac12)U(a+1,x)=0$

These are also satisfied by $\Gamma(\frac12-a)V(a,x)$.

19.6.5 $V'(a,x)-\frac12 xV(a,x)-(a-\frac12)V(a-1,x)=0$

19.6.6 $V'(a,x)+\frac12 xV(a,x)-V(a+1,x)=0$

19.6.7

$$2V'(a,x)-V(a+1,x)-(a-\tfrac12)V(a-1,x)=0$$

19.6.8

$$xV(a,x)-V(a+1,x)+(a-\tfrac12)V(a-1,x)=0$$

These are also satisfied by $U(a,x)/\Gamma(\frac12-a)$

19.6.9 $y_1'(a,x)+\frac12 xy_1(a,x)=(a+\frac12)y_2(a+1,x)$

19.6.10 $y_1'(a,x)-\frac12 xy_1(a,x)=(a-\frac12)y_2(a-1,x)$