19.3. Standard Solutions

These have been chosen to have the asymptotic behavior exhibited in **19.8.** The first is Whittaker's function [19.8, 19.9] in a more symmetrical notation.

19.3.1

$$U(a,x)=D_{-a-\frac12}(x)=\cos\pi(\tfrac14+\tfrac12a)\cdot Y_1-\sin\pi(\tfrac14+\tfrac12a)\cdot Y_2$$

19.3.2

$$V(a,x)=\frac{1}{\Gamma(\frac12-a)}\{\sin\pi(\tfrac14+\tfrac12a)\cdot Y_1+\cos\pi(\tfrac14+\tfrac12a)\cdot Y_2\}$$

in which

19.3.3 $$Y_1=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac14-\frac12a)}{2^{\frac12a+\frac14}}y_1=\sqrt{\pi}\frac{\sec\pi(\frac14+\frac12a)}{2^{\frac12a+\frac14}\Gamma(\frac34+\frac12a)}y_1$$

19.3.4 $$Y_2=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac34-\frac12a)}{2^{\frac12a-\frac14}}y_2=\sqrt{\pi}\frac{\csc\pi(\frac14+\frac12a)}{2^{\frac12a-\frac14}\Gamma(\frac14+\frac12a)}y_2$$

19.3.5

$$U(a,0)=\frac{\sqrt{\pi}}{2^{\frac12a+\frac14}\Gamma(\frac34+\frac12a)}$$

$$U'(a,0)=-\frac{\sqrt{\pi}}{2^{\frac12a-\frac14}\Gamma(\frac14+\frac12a)}$$

19.3.6

$$V(a,0)=\frac{2^{\frac12a+\frac14}\sin\pi(\frac34-\frac12a)}{\Gamma(\frac34-\frac12a)}$$

$$V'(a,0)=\frac{2^{\frac12a+\frac34}\sin\pi(\frac14-\frac12a)}{\Gamma(\frac14-\frac12a)}$$

In terms of the more familiar $D_n(x)$ of Whittaker,

19.3.7 $$U(a,x)=D_{-a-\frac12}(x)$$

19.3.8

$$V(a,x)=\frac{1}{\pi}\Gamma(\tfrac12+a)\{\sin\pi a\cdot D_{-a-\frac12}(x)+D_{-a-\frac12}(-x)\}$$

19.4. Wronskian and Other Relations

19.4.1 $$W\{U,V\}=\sqrt{2/\pi}$$

19.4.2

$$\pi V(a,x)=\Gamma(\tfrac12+a)\{\sin\pi a\cdot U(a,x)+U(a,-x)\}$$

19.4.3

$$\Gamma(\tfrac12+a)U(a,x)=\pi\sec^2\pi a\{V(a,-x)-\sin\pi a\cdot V(a,x)\}$$

19.4.4

$$\frac{\Gamma(\frac14-\frac12a)\cos\pi(\frac14+\frac12a)}{\sqrt{\pi}2^{\frac12a-\frac14}}y_1=2\sin\pi(\tfrac34+\tfrac12a)\cdot Y_1=U(a,x)+U(a,-x)$$

19.4.5

$$-\frac{\Gamma(\frac34-\frac12a)\sin\pi(\frac14+\frac12a)}{\sqrt{\pi}2^{\frac12a-\frac34}}y_2=2\cos\pi(\tfrac34+\tfrac12a)\cdot Y_2=U(a,x)-U(a,-x)$$

19.4.6

$$\sqrt{2\pi}U(-a,\pm ix)=\Gamma(\tfrac12+a)\{e^{-i\pi(\frac12a-\frac14)}U(a,\pm x)+e^{i\pi(\frac12a-\frac14)}U(a,\mp x)\}$$

19.4.7

$$\sqrt{2\pi}U(a,\pm x)=\Gamma(\tfrac12-a)\{e^{-i\pi(\frac12a+\frac14)}U(-a,\pm ix)+e^{i\pi(\frac12a+\frac14)}U(-a,\mp ix)\}$$

19.5. Integral Representations

A full treatment is given in [19.11] section 4. Representations are given here for $U(a, z)$ only; others may be derived by use of the relations given in **19.4.**

19.5.1 $$U(a,z)=\frac{\Gamma(\frac12-a)}{2\pi i}e^{-\frac14z^2}\int_\alpha e^{zs-\frac12s^2}s^{a-\frac12}ds$$

19.5.2 $$=\frac{\Gamma(\frac12-a)}{2\pi i}e^{\frac14z^2}\int_\beta e^{-\frac12t^2}(z+t)^{a-\frac12}dt$$

where α and β are the contours shown in **Figures 19.1 and 19.2.**

When $a+\frac12$ is a positive integer these integrals become indeterminate; in this case

19.5.3 $$U(a,z)=\frac{1}{\Gamma(\frac12+a)}e^{-\frac14z^2}\int_0^\infty e^{-zs-\frac12s^2}s^{a-\frac12}ds$$

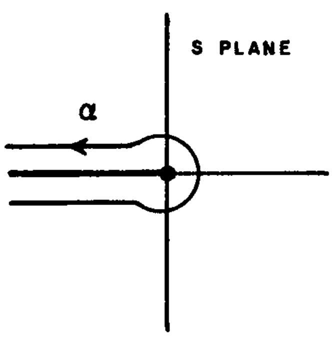

FIGURE 19.1

$-\pi<\arg s<\pi$

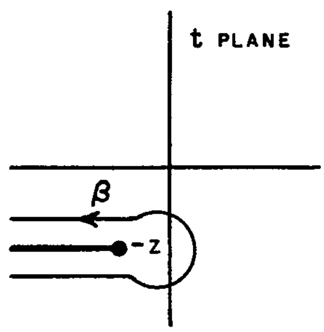

FIGURE 19.2

$-\pi<\arg (z+t)>\pi$