4.1.23 $\ln z = \ln 10 \log_{10} z = (2.30258\ 50929\ldots)\log_{10} z$

($\log_e x = \ln x$, called natural, Napierian, or hyperbolic logarithms; $\log_{10} x$, called common or Briggs logarithms.)

Series Expansions

4.1.24 $$\ln(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \ldots \qquad (|z|\le 1 \text{ and } z\ne -1)$$

4.1.25 $$\ln z = \left(\frac{z-1}{z}\right) + \frac{1}{2}\left(\frac{z-1}{z}\right)^2 + \frac{1}{3}\left(\frac{z-1}{z}\right)^3 + \ldots \qquad (\mathscr{R}z \ge \tfrac{1}{2})$$

4.1.26 $$\ln z = (z-1) - \tfrac{1}{2}(z-1)^2 + \tfrac{1}{3}(z-1)^3 - \ldots \qquad (|z-1|\le 1,\quad z\ne 0)$$

4.1.27 $$\ln z = 2\left[\left(\frac{z-1}{z+1}\right) + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \ldots\right] \qquad (\mathscr{R}z \ge 0,\quad z\ne 0)$$

4.1.28 $$\ln\left(\frac{z+1}{z-1}\right) = 2\left(\frac{1}{z} + \frac{1}{3z^3} + \frac{1}{5z^5} + \ldots\right) \qquad (|z|\ge 1,\quad z\ne \pm 1)$$

4.1.29 $$\ln(z+a) = \ln a + 2\left[\left(\frac{z}{2a+z}\right) + \frac{1}{3}\left(\frac{z}{2a+z}\right)^3 + \frac{1}{5}\left(\frac{z}{2a+z}\right)^5 + \ldots\right] \qquad (a>0,\quad \mathscr{R}z \ge -a \ne z)$$

Limiting Values

4.1.30 $$\lim_{x\to\infty} x^{-\alpha}\ln x = 0 \qquad (\alpha \text{ constant},\quad \mathscr{R}\alpha > 0)$$

4.1.31 $$\lim_{x\to 0} x^{\alpha}\ln x = 0 \qquad (\alpha \text{ constant},\quad \mathscr{R}\alpha > 0)$$

4.1.32 $$\lim_{m\to\infty}\left(\sum_{k=1}^{m}\frac{1}{k} - \ln m\right) = \gamma \text{ (Euler's constant)} = .57721\ 56649\ldots$$

(see chapters **1**, **6** and **23**)

Inequalities

4.1.33 $$\frac{x}{1+x} < \ln(1+x) < x \qquad (x>-1,\quad x\ne 0)$$

4.1.34 $$x < -\ln(1-x) < \frac{x}{1-x} \qquad (x<1,\quad x\ne 0)$$

4.1.35 $$|\ln(1-x)| < \frac{3x}{2} \qquad (0 < x \le .5828)$$

4.1.36 $$\ln x \le x-1 \qquad (x>0)$$

4.1.37 $\ln x \le n(x^{1/n}-1)$ for any positive n $\qquad (x>0)$

4.1.38 $$|\ln(1+z)| \le -\ln(1-|z|) \qquad (|z|<1)$$

Continued Fractions

4.1.39 $$\ln(1+z) = \frac{z}{1+}\,\frac{z}{2+}\,\frac{z}{3+}\,\frac{4z}{4+}\,\frac{4z}{5+}\,\frac{9z}{6+}\cdots$$

(z in the plane cut from -1 to $-\infty$)

4.1.40 $$\ln\left(\frac{1+z}{1-z}\right) = \frac{2z}{1-}\,\frac{z^2}{3-}\,\frac{4z^2}{5-}\,\frac{9z^2}{7-}\cdots$$

(z in the cut plane of Figure 4.7.)

Polynomial Approximations [2]

4.1.41 $$\frac{1}{\sqrt{10}} \le x \le \sqrt{10}$$

$$\log_{10} x = a_1 t + a_3 t^3 + \epsilon(x), \quad t = (x-1)/(x+1)$$

$$|\epsilon(x)| \le 6\times 10^{-4}$$

$a_1 = .86304 \qquad a_3 = .36415$

4.1.42 $$\frac{1}{\sqrt{10}} \le x \le \sqrt{10}$$

$$\log_{10} x = a_1 t + a_3 t^3 + a_5 t^5 + a_7 t^7 + a_9 t^9 + \epsilon(x)$$

$$t = (x-1)/(x+1)$$

$$|\epsilon(x)| \le 10^{-7}$$

$a_1 = .86859\ 1718 \qquad a_7 = .09437\ 6476$

$a_3 = .28933\ 5524 \qquad a_9 = .19133\ 7714$

$a_5 = .17752\ 2071$

4.1.43 $$0 \le x \le 1$$

$$\ln(1+x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \epsilon(x)$$

$$|\epsilon(x)| \le 1\times 10^{-5}$$

$a_1 = .99949\ 556 \qquad a_4 = -.13606\ 275$

$a_2 = -.49190\ 896 \qquad a_5 = .03215\ 845$

$a_3 = .28947\ 478$

[2] The approximations **4.1.41** to **4.1.44** are from C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N.J., 1955 (with permission).