$$m=\sin^2\alpha=4/5,\ \lambda=\sqrt{5}/2,\ \cos^2\varphi=3/4.$$

Thus $\alpha=63.434949°$, $\varphi=30°$ and

$$u=2(5)^{-\frac{1}{4}}F(30°\backslash 63.434949°)$$
$$=2(5)^{-\frac{1}{4}}(.543604)=.486214 \text{ from } \textbf{Table 17.5.}$$

The above integral is of the Weierstrass type and in fact $17=\wp(\frac{1}{2}u;\ 28,\ -24)$ (see chapter **18**).

Example 11. Evaluate

$$\int_0^{2/3}(24-12t+2t^2-t^3)^{-1/2}\,dt.$$

We have

$$24-12t+2t^2-t^3=-(t-2)(t^2+12)=-P(t).$$

There is only one real zero and we therefore use **17.4.74** with $P(t)=t^3-2t^2+12t-24$, $\beta=2$ so that $P'(2)=16$, $P''(2)=8$, $\lambda=2$ and therefore

$$m=\sin^2\alpha=\frac{1}{4},\quad \alpha=30°.$$

Therefore the given integral is

$$\int_0^2-\int_{2/3}^2=\frac{1}{2}\,[F(\varphi_1\backslash 60°)-F(\varphi_2\backslash 60°)]$$

where

$$\cos\varphi_1=\frac{1}{3},\qquad \varphi_1=70.52877\ 93°$$

$$\cos\varphi_2=\frac{1}{2},\qquad \varphi_2=60°$$

and the integral $=\frac{1}{2}[1.510344-1.212597]=.148874.$

Example 12. Use Landen's transformation to evaluate

$$\int_0^{\pi/2}\left(1-\frac{1}{4}\sin^2\theta\right)^{-1/2}d\theta \text{ to 5D.}$$

First Method, Descending Transformation

We use **17.5.1** to give

$$1+\sin\alpha_1=\frac{2}{1+\cos 30°}=1.071797$$

$$\cos\alpha_1=[(1-\sin\alpha_1)(1+\sin\alpha_1)]^{1/2}=.997419$$

$$1+\sin\alpha_2=\frac{2}{1+\cos\alpha_1}=1.001292;\ \cos\alpha_2=.999999$$

$$1+\sin\alpha_3=\frac{2}{1+\cos\alpha_2}=1.000000$$

Thus from **17.5.7,**

$$\text{the integral}=F(90°\backslash 30°)=\frac{\pi}{2}\,(1.071797)(1.001292)$$
$$=1.68575 \text{ to 5D.}$$

Second Method, Ascending Transformation

We use **17.5.11** to give

$$1+\cos\alpha_{n+1}=2/(1+\sin\alpha_n)$$

n	$\cos\alpha_n$	$\sin\alpha_n$
1	.33333 333	.94280 904
2	.02943 725	.99956 663
3	.00021 673	.99999 998

$$\sin(2\varphi_1-90°)=\sin 30°,\qquad \varphi_1=60°$$
$$\sin(2\varphi_2-\varphi_1)=\sin\alpha_1\sin\varphi_1,\qquad \varphi_2=57.367805°$$
$$\sin(2\varphi_3-\varphi_2)=\sin\alpha_2\sin\varphi_2,\qquad \varphi_3=57.348426°$$
$$\sin(2\varphi_4-\varphi_3)=\sin\alpha_3\sin\varphi_3,\qquad \varphi_4=57.348425°=\Phi.$$

From **17.5.16**

$$F(90°\backslash 30°)=\frac{2}{1.5}\,\frac{2}{1.94280\ 904}\,\frac{2}{1.99956\ 663}$$
$$\frac{2}{1.99999\ 998}\ln\tan\left(45°+\frac{1}{2}\Phi\right)$$
$$=1.37288\ 050\ln\tan 73.674213°$$
$$=1.37288\ 050(1.22789\ 30)$$

$F(90°\backslash 30°)=1.68575$ to 5D.

Example 13. Find the value of $F(89.5°\backslash 89.5°)$.

First Method

This is a case where interpolation in **Table 17.5** is not possible. We use **17.4.13** which gives

$$F(89.5°\backslash 89.5°)=F(90°\backslash 89.5°)-F(\psi\backslash 89.5°)$$

where

$$\cot\psi=\sin(.5°)\cot(.5°)=\cos(.5°)$$
$$\psi=45.00109\ 084°$$

and $F(\psi\backslash 89.5°)=.881390$ from **Table 17.5.**

$$F(90°\backslash 89.5°)=K(\sin^2 89.5°)=K(.99992\ 38476)$$
$$=6.12777\ 88$$

Thus $F(89.5°\backslash 89.5°)=5.246389.$

Second Method

Landen's ascending transformation, **17.5.11,** gives