

If the quartic $y^2=0$ has four real roots in x (or in the case of a cubic all three roots are real), we must so combine the factors that no root of $Q_1=0$ lies between the roots of $Q_2=0$ and no root of $Q_2=0$ lies between the roots of $Q_1=0$. Provided this condition is observed the method just described will always lead to real values of λ . These values may, however, be irrational.

Second Method

Write

$$t^2 = \frac{Q_1}{Q_2} = \frac{3x^2 - 10x + 9}{-x^2 + 8x - 10}$$

and let the discriminant of $Q_2 t^2 - Q_1$ be

$$4T^2 = (8t^2 + 10)^2 - 4(t^2 + 3)(10t^2 + 9) \\ = 4(3t^2 + 2)(2t^2 - 1)$$

Then

$$\int y^{-1} dx = \pm \int T^{-1} dt = \pm \int [(3t^2 + 2)(2t^2 - 1)]^{-1/2} dt$$

This method will succeed if, as here, T^2 as a function of t^2 has real factors. If the coefficients of the given quartic are rational numbers, the factors of T^2 will likewise be rational.

Third Method

Write

$$w = \frac{Q_1}{Q_2} = \frac{3x^2 - 10x + 9}{-x^2 + 8x - 10}$$

and let the discriminant of $Q_2 w - Q_1$ be

$$4W = 4(3w + 2)(2w - 1) = 4(Aw^2 + Bw + C)$$

Then if

$$z^2 = W/w \text{ and } Z^2 = (B - z^2)^2 - 4AC = (z^2 - 1)^2 + 48$$

$$\int y^{-1} dx = \pm \int Z^{-1} dz$$

However, in this case the factors of Z are complex and the method fails.

Of the second and third methods one will always succeed where the other fails, and if the coefficients of the given quartic are rational numbers, the factors of T^2 or Z^2 , as the case may be, will be rational.

Example 2. Reduce to canonical form $\int y^{-1} dx$ where $y^2 = x(x-1)(x-2)$.

We use the third method of **Example 1** taking $Q_1 = (x-1)$, $Q_2 = x(x-2)$ and writing

$$w = \frac{Q_1}{Q_2} = \frac{x-1}{x^2-2x}$$

The discriminant of $Q_2 w - Q_1 = x^2 w - (2w+1)x + 1$ is

$$4W = (2w+1)^2 - 4w = 4w^2 + 1$$

so that

$$W = Aw^2 + Bw + C \text{ where } A=1, B=0, C=\frac{1}{4}$$

and if we write $z^2 = W/w$ and

$$Z^2 = (B - z^2)^2 - 4AC = (z^2)^2 - 1 = (z^2 - 1)(z^2 + 1),$$

$$\int y^{-1} dx = \pm \int [(z^2 - 1)(z^2 + 1)]^{-1/2} dz$$

The first method of **Example 1** fails with the above values of Q_1 and Q_2 since the root of $Q_1=0$ lies between the roots of $Q_2=0$, and we get imaginary values of λ . The method succeeds, however, if we take $Q_1 = x$, $Q_2 = (x-1)(x-2)$, for then the roots of $Q_1=0$ do not lie between those of $Q_2=0$.

Example 3. Find $K(80/81)$.

First Method

Use 17.3.29 with $m=80/81$, $m_1=1/81$, $m_1^{1/2}=1/9$. Since $[(1 - m_1^{1/2})(1 + m_1^{1/2})^{-1}]^2 = .64$, $K(80/81) = 1.8 K(.64) = 3.59154 \text{ 500 to 8D}$, taking $K(.64)$ from **Table 17.1**.

Second Method

Table 17.4 giving $L(m)$ is useful for computing $K(m)$ when m is near unity or $K'(m)$ when m is near zero.

$$K(80/81) = \frac{1}{\pi} K'(80/81) \ln(16 \times 81) - L(80/81).$$

By interpolation in **Tables 17.1** and **17.4**, since $80/81 = .98765 \text{ 43210}$,

$$K'(80/81) = 1.57567 \text{ 8423}$$

$$L(80/81) = .00311 \text{ 16543}$$

$$K(80/81) = \pi^{-1}(1.57567 \text{ 8423})(7.16703 \text{ 7877})$$

$$- .00311 \text{ 16543}$$

$$= 3.59154 \text{ 5000 to 9D.}$$

Third Method

The polynomial approximation 17.3.34 gives to 8D

$$K(80/81) = 3.59154 \text{ 501}$$

Fourth Method, Arithmetic-Geometric Mean

Here $\sin^2 \alpha = 80/81$ and we start with

$$a_0 = 1, b_0 = \frac{1}{9}, c_0 = \sqrt{80/81} = .99380 \text{ 79900}$$

giving