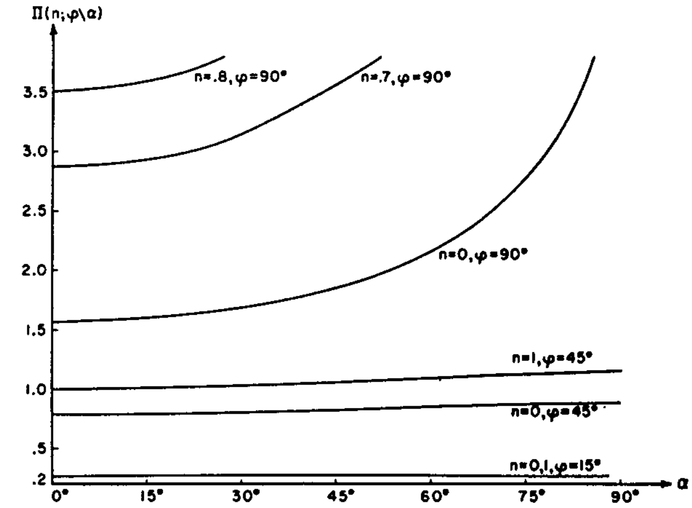

FIGURE 17.11. *Elliptic integral of the third kind* $\Pi(n;\,\varphi\backslash\alpha)$.

Case (iv) Circular Case $n<0$

The case $n<0$ can be reduced to the case $\sin^2\alpha<N<1$ by writing

17.7.15

$$N=(\sin^2\alpha-n)(1-n)^{-1}$$

$$p_2=[-n(1-n)^{-1}(\sin^2\alpha-n)]^{\frac{1}{2}}$$

17.7.16

$$[(1-n)(1-n^{-1}\sin^2\alpha)]^{\frac{1}{2}}\Pi(n;\,\varphi\backslash\alpha)$$
$$=[(1-N)(1-N^{-1}\sin^2\alpha)]^{\frac{1}{2}}\Pi(N;\,\varphi\backslash\alpha)$$
$$+p_2^{-1}\sin^2\alpha F(\varphi\backslash\alpha)+\arctan\,[\tfrac{1}{2}p_2\sin 2\varphi/\Delta(\varphi)]$$

17.7.17

$$\Pi(n\backslash\alpha)=(-n\cos^2\alpha)(1-n)^{-1}(\sin^2\alpha-n)^{-1}\Pi(N\backslash\alpha)$$
$$+\sin^2\alpha(\sin^2\alpha-n)^{-1}K(\alpha)$$

Special Cases

17.7.18 $n=0$

$$\Pi(0;\,\varphi\backslash\alpha)=F(\varphi\backslash\alpha)$$

17.7.19 $n=0,\ \alpha=0$

$$\Pi(0;\,\varphi\backslash 0)=\varphi$$

17.7.20 $\alpha=0$

$$\Pi(n;\,\varphi\backslash 0)=(1-n)^{-\frac{1}{2}}\arctan\,[(1-n)^{\frac{1}{2}}\tan\varphi],\quad n<1 \quad *$$
$$=(n-1)^{-\frac{1}{2}}\operatorname{arctanh}\,[(n-1)^{\frac{1}{2}}\tan\varphi],\quad n>1$$
$$=\tan\varphi\qquad n=1$$

17.7.21 $\alpha=\pi/2$

$$\Pi(n;\,\varphi\backslash\pi/2)=(1-n)^{-1}[\ln\,(\tan\varphi+\sec\varphi)$$
$$-\tfrac{1}{2}n^{\frac{1}{2}}\ln\,(1+n^{\frac{1}{2}}\sin\varphi)(1-n^{\frac{1}{2}}\sin\varphi)^{-1}]\qquad n\neq 1$$

17.7.22 $n=\pm\sin\alpha$

$$(1\mp\sin\alpha)\{2\Pi(\pm\sin\alpha;\,\varphi\backslash\alpha)-F(\varphi\backslash\alpha)\}$$
$$=\arctan\,[(1\mp\sin\alpha)\tan\varphi/\Delta(\varphi)]$$

17.7.23 $n=1\pm\cos\alpha$

$$2\cos\alpha\Pi(1\pm\cos\alpha;\,\varphi\backslash\alpha)=\pm\tfrac{1}{2}\ln\,[(1+\tan\varphi$$
$$\cdot\Delta(\varphi))(1-\tan\varphi\cdot\Delta(\varphi))^{-1}]+\tfrac{1}{2}\ln\,[(\Delta(\varphi)$$
$$+\cos\alpha\cdot\tan\varphi)(\Delta(\varphi)-\cos\alpha\tan\varphi)^{-1}]$$
$$\mp(1\mp\cos\alpha)F(\varphi\backslash\alpha)$$

17.7.24 $n=\sin^2\alpha$

$$\Pi(\sin^2\alpha;\,\varphi\backslash\alpha)=\sec^2\alpha E(\varphi\backslash\alpha)-(\tan^2\alpha\sin 2\varphi)/(2\Delta(\varphi))$$

17.7.25 $n=1$

$$\Pi(1;\,\varphi\backslash\alpha)=F(\varphi\backslash\alpha)-\sec^2\alpha E(\varphi\backslash\alpha)+\sec^2\alpha\tan\varphi\Delta(\varphi)$$

Numerical Methods

17.8. Use and Extension of the Tables

Example 1. Reduce to canonical form $\int y^{-1}dx$, where

$$y^2=-3x^4+34x^3-119x^2+172x-90$$

By inspection or by solving an equation of the fourth degree we find that

$$y^2=Q_1Q_2 \text{ where } Q_1=3x^2-10x+9,\ Q_2=-x^2+8x-10$$

First Method

$Q_1-\lambda Q_2=(3+\lambda)x^2-(10+8\lambda)x+9+10\lambda$ is a perfect square if the discriminant

$(10+8\lambda)^2-4(3+\lambda)(9+10\lambda)=0$; i.e., if $\lambda=-\frac{2}{3}$ or $\frac{1}{2}$ and then

$$Q_1+\frac{2}{3}Q_2=\frac{7}{3}(x-1)^2,\ Q_1-\frac{1}{2}Q_2=\frac{7}{2}(x-2)^2$$

Solving for Q_1 and Q_2 we get

$$Q_1=(x-1)^2+2(x-2)^2,\ Q_2=2(x-1)^2-3(x-2)^2$$

The substitution $t=(x-1)/(x-2)$ then gives

$$\int y^{-1}dx=\pm\int[(t^2+2)(2t^2-3)]^{-\frac{1}{2}}dt$$

*See page II.