Jacobi's Zeta Function

17.4.27 $Z(\varphi\backslash\alpha)=E(\varphi\backslash\alpha)-E(\alpha)F(\varphi\backslash\alpha)/K(\alpha)$

17.4.28 $Z(u|m)=Z(u)=E(u)-uE(m)/K(m)$

17.4.29 $Z(-u)=-Z(u)$

17.4.30 $Z(u+2K)=Z(u)$

17.4.31 $Z(K-u)=-Z(K+u)$

17.4.32 $Z(u)=Z(u-K)-m\,\mathrm{sn}(u-K)\mathrm{cd}(u-K)$

Special Values

17.4.33 $Z(u|0)=0$

17.4.34 $Z(u|1)=\tanh u$

Addition Theorem

17.4.35

$$Z(u+v)=Z(u)+Z(v)-m\,\mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{sn}(u+v)$$

Jacobi's Imaginary Transformation

17.4.36

$$iZ(iu|m)=Z(u|m_1)+\frac{\pi u}{2KK'}-\mathrm{dn}(u|m_1)\mathrm{sc}(u|m_1)$$

Relation to Jacobi's Theta Function

17.4.37 $Z(u)=\Theta'(u)/\Theta(u)=\frac{d}{du}\ln\Theta(u)$

q-Series

17.4.38 $Z(u)=\frac{2\pi}{K}\sum_{s=1}^{\infty}q^s(1-q^{2s})^{-1}\sin(\pi su/K)$

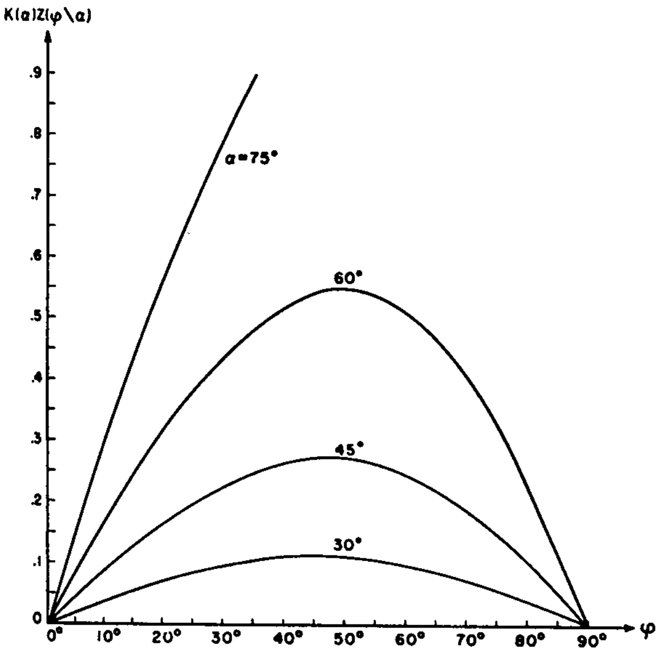

FIGURE 17.9. *Jacobian zeta function* $K(\alpha)Z(\varphi\backslash\alpha)$.

*See page II.

Heuman's Lambda Function

17.4.39

$$\Lambda_0(\varphi\backslash\alpha)=\frac{F(\varphi\backslash 90^\circ-\alpha)}{K'(\alpha)}+\frac{2}{\pi}K(\alpha)Z(\varphi\backslash 90^\circ-\alpha)$$

17.4.40

$$=\frac{2}{\pi}\{K(\alpha)E(\varphi\backslash 90^\circ-\alpha)-[K(\alpha)-E(\alpha)]F(\varphi\backslash 90^\circ-\alpha)\}$$

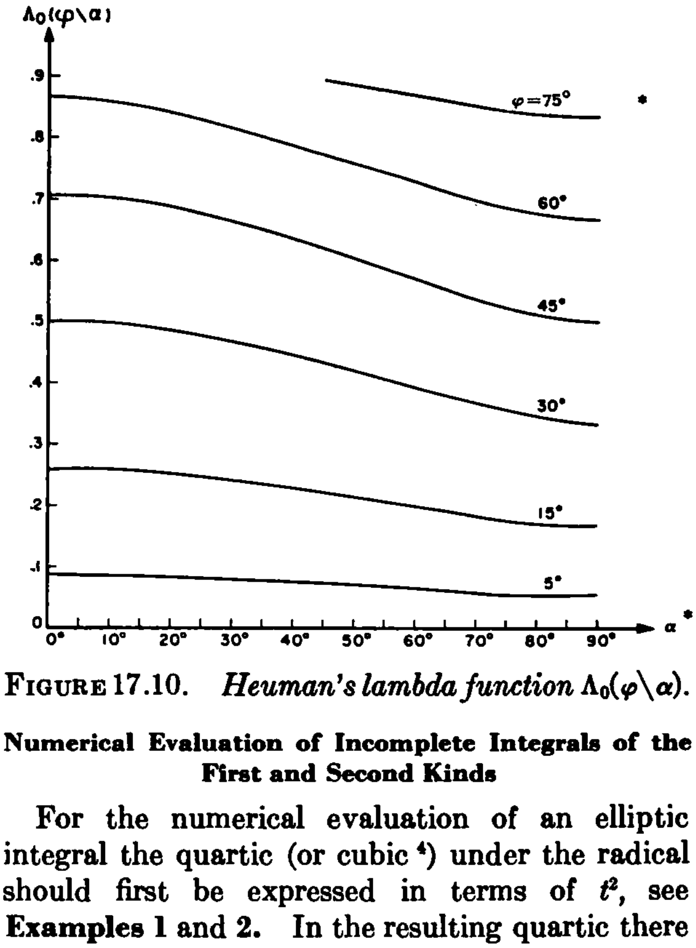

FIGURE 17.10. *Heuman's lambda function* $\Lambda_0(\varphi\backslash\alpha)$.

Numerical Evaluation of Incomplete Integrals of the First and Second Kinds

For the numerical evaluation of an elliptic integral the quartic (or cubic [4]) under the radical should first be expressed in terms of t^2, see **Examples 1 and 2.** In the resulting quartic there are only six possible sign patterns or combinations of the factors namely

$$(t^2+a^2)(t^2+b^2),\quad (a^2-t^2)(t^2-b^2),$$
$$(a^2-t^2)(b^2-t^2),\quad (t^2-a^2)(t^2-b^2),\quad (t^2+a^2)(t^2-b^2),$$
$$(t^2+a^2)(b^2-t^2).$$

The list which follows is then exhaustive for integrals which reduce to $F(\varphi\backslash\alpha)$ or $E(\varphi\backslash\alpha)$.

The value of the elliptic integral of the first kind is also expressed as an *inverse* Jacobian elliptic function. Here, for example, the notation $u=\mathrm{sn}^{-1}x$ means that $x=\mathrm{sn}\,u$.

The column headed "t substitution" gives the Jacobian elliptic function substitution which is appropriate to reduce every elliptic integral which contains the given quartic.

[4] For an alternate treatment of cubics see **17.4.61** and **17.4.70.**