Relation to the Hypergeometric Function
(see chapter 15)

17.3.9 $K=\frac{1}{2}\pi F(\frac{1}{2},\frac{1}{2};1;m)$

17.3.10 $E=\frac{1}{2}\pi F(-\frac{1}{2},\frac{1}{2};1;m)$

Infinite Series

17.3.11

$$K(m)=\frac{1}{2}\pi\left[1+\left(\frac{1}{2}\right)^2 m+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 m^2+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 m^3+\ldots\right] \quad (|m|<1)$$

17.3.12

$$E(m)=\frac{1}{2}\pi\left[1-\left(\frac{1}{2}\right)^2 \frac{m}{1}-\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{m^2}{3}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{m^3}{5}-\ldots\right] \quad (|m|<1)$$

Legendre's Relation

17.3.13 $EK'+E'K-KK'=\frac{1}{2}\pi$

Auxiliary Function

17.3.14 $L(m)=\frac{K'(m)}{\pi}\ln\frac{16}{m_1}-K(m)$

17.3.15 $m=1-16\exp\left[-\pi(K(m)+L(m))/K'(m)\right]$

17.3.16 $m=16\exp\left[-\pi(K'(m)+L(m_1))/K(m)\right]$

The function $L(m)$ is tabulated in **Table 17.4.**

q-Series

The Nome q and the Complementary Nome q_1

17.3.17 $q=q(m)=\exp\left[-\pi K'/K\right]$

17.3.18 $q_1=q(m_1)=\exp\left[-\pi K/K'\right]$

17.3.19 $\ln\frac{1}{q}\ln\frac{1}{q_1}=\pi^2$

17.3.20

$$\log_{10}\frac{1}{q}\log_{10}\frac{1}{q_1}=(\pi\log_{10}e)^2=1.86152\ 28349 \text{ to 10D}$$

17.3.21

$$q=\exp\left[-\pi K'/K\right]=\frac{m}{16}+8\left(\frac{m}{16}\right)^2+84\left(\frac{m}{16}\right)^3+992\left(\frac{m}{16}\right)^4+\ldots \quad (|m|<1)$$

17.3.22 $K=\frac{1}{2}\pi+2\pi\sum_{s=1}^{\infty}\frac{q^s}{1+q^{2s}}$

17.3.23

$$\frac{E}{K}=\frac{1}{3}(1+m_1)+(\pi/K)^2\left[1/12-2\sum_{s=1}^{\infty}q^{2s}(1-q^{2s})^{-2}\right]$$

17.3.24 $\operatorname{am} u=v+\sum_{s=1}^{\infty}\frac{2q^s\sin 2sv}{s(1+q^{2s})}$ where $v=\pi u/(2K)$

Limiting Values

17.3.25 $\lim_{m\to 0}K'(E-K)=0$

17.3.26 $\lim_{m\to 1}\left[K-\frac{1}{2}\ln(16/m_1)\right]=0$

17.3.27 $\lim_{m\to 0}m^{-1}(K-E)=\lim_{m\to 0}m^{-1}(E-m_1K)=\pi/4$

17.3.28 $\lim_{m\to 0}q/m=\lim_{m_1\to 1}q_1/m_1=1/16$

Alternative Evaluations of K and E (see also 17.5)

17.3.29

$$K(m)=2[1+m_1^{1/2}]^{-1}K([(1-m_1^{1/2})/(1+m_1^{1/2})]^2)^*$$

17.3.30

$$E(m)=(1+m_1^{1/2})E([(1-m_1^{1/2})/(1+m_1^{1/2})]^2)-2m_1^{1/2}(1+m_1^{1/2})^{-1}K([(1-m_1^{1/2})/(1+m_1^{1/2})]^2)$$

17.3.31 $K(\alpha)=2F(\arctan(\sec^{1/2}\alpha)\backslash\alpha)$

17.3.32 $E(\alpha)=2E(\arctan(\sec^{1/2}\alpha)\backslash\alpha)-1+\cos\alpha$

Polynomial Approximations[3] ($0\le m<1$)

17.3.33

$$K(m)=[a_0+a_1m_1+a_2m_1^2]+[b_0+b_1m_1+b_2m_1^2]\ln(1/m_1)+\epsilon(m)$$

$$|\epsilon(m)|\le 3\times 10^{-5}$$

$a_0=1.38629\ 44$	$b_0=.5$
$a_1=\ .11197\ 23$	$b_1=.12134\ 78$
$a_2=\ .07252\ 96$	$b_2=.02887\ 29$

17.3.34

$$K(m)=[a_0+a_1m_1+\ldots+a_4m_1^4]+[b_0+b_1m_1+\ldots+b_4m_1^4]\ln(1/m_1)+\epsilon(m)$$

$$|\epsilon(m)|\le 2\times 10^{-8}$$

$a_0=1.38629\ 436112$	$b_0=.5$
$a_1=\ .09666\ 344259$	$b_1=.12498\ 593597$
$a_2=\ .03590\ 092383$	$b_2=.06880\ 248576$
$a_3=\ .03742\ 563713$	$b_3=.03328\ 355346$
$a_4=\ .01451\ 196212$	$b_4=.00441\ 787012$

[3] The approximations **17.3.33–17.3.36** are from C. Hastings, Jr., Approximations for Digital Computers, Princeton Univ. Press, Princeton, N. J. (with permission).

*See page II.