15.3.2 $$F(a,b;c;z)=\frac{\Gamma(c)}{2\pi i\Gamma(a)\Gamma(b)}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds$$

$$=\tfrac{1}{2}i\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(1+s)\Gamma(c+s)}\csc(\pi s)(-z)^s ds$$

Here $-\pi<\arg(-z)<\pi$ and the path of integration is chosen such that the poles of $\Gamma(a+s)$ and $\Gamma(b+s)$ i.e. the points $s=-a-n$ and $s=-b-m(n, m=0, 1, 2, \ldots)$ respectively, are at its left side and the poles of $\csc(\pi s)$ or $\Gamma(-s)$ i.e. $s=0, 1, 2,$ are at its right side. The cases in which $-a$, $-b$ or $-c$ are non-negative integers or $a-b$ equal to an integer are excluded.

Linear Transformation Formulas

From **15.3.1** and **15.3.2** a number of transformation formulas for $F(a, b; c; z)$ can be derived.

15.3.3 $$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z)$$

15.3.4 $$=(1-z)^{-a}F\left(a,c-b;c;\frac{z}{z-1}\right)$$

15.3.5 $$=(1-z)^{-b}F\left(b,c-a;c;\frac{z}{z-1}\right)$$

15.3.6 $$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-z)$$
$$+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b;c-a-b+1;1-z)$$
$$(|\arg(1-z)|<\pi)$$

15.3.7 $$=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F\left(a,1-c+a;1-b+a;\frac{1}{z}\right)$$
$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F\left(b,1-c+b;1-a+b;\frac{1}{z}\right)\qquad(|\arg(-z)|<\pi)$$

15.3.8 $$=(1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}F\left(a,c-b;a-b+1;\frac{1}{1-z}\right)$$
$$+(1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}F\left(b,c-a;b-a+1;\frac{1}{1-z}\right)\qquad(|\arg(1-z)|<\pi)$$

15.3.9 $$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}z^{-a}F\left(a,a-c+1;a+b-c+1;1-\frac{1}{z}\right)$$
$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}z^{a-c}F\left(c-a,1-a;c-a-b+1;1-\frac{1}{z}\right)$$
$$(|\arg z|<\pi,\ |\arg(1-z)|<\pi)$$

Each term of **15.3.6** has a pole when $c=a+b\pm m$, $(m=0, 1, 2, \ldots)$; this case is covered by

15.3.10 $$F(a,b;a+b;z)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(n!)^2}[2\psi(n+1)-\psi(a+n)-\psi(b+n)-\ln(1-z)](1-z)^n$$
$$(|\arg(1-z)|<\pi,\ |1-z|<1)$$

Furthermore for $m=1, 2, 3, \ldots$

15.3.11 $$F(a,b;a+b+m;z)=\frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)}\sum_{n=0}^{m-1}\frac{(a)_n(b)_n}{n!(1-m)_n}(1-z)^n$$
$$-\frac{\Gamma(a+b+m)}{\Gamma(a)\Gamma(b)}(z-1)^m\sum_{n=0}^{\infty}\frac{(a+m)_n(b+m)_n}{n!(n+m)!}(1-z)^n[\ln(1-z)-\psi(n+1)$$
$$-\psi(n+m+1)+\psi(a+n+m)+\psi(b+n+m)]\qquad(|\arg(1-z)|<\pi,|1-z|<1)$$