13.2.8 $\Gamma(a)U(a,b,z)$

$$=e^{Az}\int_A^\infty e^{-zt}(t-A)^{a-1}(t+B)^{b-a-1}dt \qquad (A=1-B)$$

Similar integrals for $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ can be deduced with the help of 13.1.32 and 13.1.33.

Barnes-type Contour Integrals

13.2.9

$$\frac{\Gamma(a)}{\Gamma(b)}M(a,b,z)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(-s)\Gamma(a+s)}{\Gamma(b+s)}(-z)^s ds$$

for $|\arg(-z)|<\frac{1}{2}\pi$, $a, b\neq 0, -1, -2, \ldots$. The contour must separate the poles of $\Gamma(-s)$ from those of $\Gamma(a+s)$; c is finite.

13.2.10

$\Gamma(a)\Gamma(1+a-b)z^aU(a,b,z)$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(-s)\Gamma(a+s)\Gamma(1+a-b+s)z^{-s}ds$$

for $|\arg z|<\frac{3\pi}{2}$, $a\neq 0, -1, -2, \ldots, b-a\neq 1, 2, 3, \ldots$. The contour must separate the poles of $\Gamma(-s)$ from those of $\Gamma(a+s)$ and $\Gamma(1+a-b+s)$.

13.3. Connections With Bessel Functions (see chapters 9 and 10)

Bessel Functions as Limiting Cases

If b and z are fixed,

13.3.1 $\lim_{a\to\infty}\{M(a,b,z/a)/\Gamma(b)\}=z^{\frac{1}{2}-\frac{1}{2}b}I_{b-1}(2\sqrt{z})$

13.3.2 $\lim_{a\to\infty}\{M(a,b,-z/a)/\Gamma(b)\}=z^{\frac{1}{2}-\frac{1}{2}b}J_{b-1}(2\sqrt{z})$

13.3.3

$$\lim_{a\to\infty}\{\Gamma(1+a-b)\,U(a,b,z/a)\}=2z^{\frac{1}{2}-\frac{1}{2}b}K_{b-1}(2\sqrt{z})$$

13.3.4

$\lim_{a\to\infty}\{\Gamma(1+a-b)U(a,b,-z/a)\}$

$$=-\pi i e^{\pi i b}z^{\frac{1}{2}-\frac{1}{2}b}H^{(1)}_{b-1}(2\sqrt{z}) \qquad (\mathscr{I}z>0)$$

13.3.5 $$=\pi i e^{-\pi i b}z^{\frac{1}{2}-\frac{1}{2}b}H^{(2)}_{b-1}(2\sqrt{z}) \qquad (\mathscr{I}z<0)$$

Expansions in Series

13.3.6

$M(a,b,z)=e^{\frac{1}{2}z}\Gamma(b-a-\frac{1}{2})(\frac{1}{4}z)^{a-b+\frac{1}{2}}$

* $$\cdot\sum_{n=0}^{\infty}\frac{(2b-2a-1)_n(b-2a)_n(b-a-\frac{1}{2}+n)}{n!(b)_n}(-1)^n I_{b-a-\frac{1}{2}+n}(\tfrac{1}{2}z)\ (b\neq 0,-1,-2,\ldots)$$

13.3.7

$$\frac{M(a,b,z)}{\Gamma(b)}=e^{\frac{1}{2}z}(\tfrac{1}{2}bz-az)^{\frac{1}{2}-\frac{1}{2}b}\cdot\sum_{n=0}^{\infty}A_n(\tfrac{1}{2}z)^{\frac{1}{2}n}(b-2a)^{-\frac{1}{2}n}J_{b-1+n}(\sqrt{(2zb-4za)})$$

where

$A_0=1, A_1=0, A_2=\frac{1}{2}b,$

$$(n+1)A_{n+1}=(n+b-1)A_{n-1}+(2a-b)A_{n-2}, \qquad (a \text{ real})$$

13.3.8

$$\frac{M(a,b,z)}{\Gamma(b)}=e^{hz}\sum_{n=0}^{\infty}C_nz^n(-az)^{\frac{1}{2}(1-b-n)}J_{b-1+n}(2\sqrt{(-az)})$$

where

$C_0=1,\ C_1=-bh,\ C_2=-\frac{1}{2}(2h-1)a+\frac{1}{2}b(b+1)h^2,$

$$(n+1)C_{n+1}=[(1-2h)n-bh]C_n+[(1-2h)a-h(h-1)(b+n-1)]C_{n-1}-h(h-1)aC_{n-2} \qquad (h \text{ real})$$

13.3.9 $M(a,b,z)=\sum_{n=0}^{\infty}C_n(a,b)I_n(z)$

where

$C_0=1,\ C_1(a,b)=2a/b,$

$$C_{n+1}(a,b)=2aC_n(a+1,b+1)/b-C_{n-1}(a,b)$$

13.4. Recurrence Relations and Differential Properties

13.4.1

$$(b-a)M(a-1,b,z)+(2a-b+z)M(a,b,z)-aM(a+1,b,z)=0$$

13.4.2

$$b(b-1)M(a,b-1,z)+b(1-b-z)M(a,b,z)+z(b-a)M(a,b+1,z)=0$$

13.4.3

$$(1+a-b)M(a,b,z)-aM(a+1,b,z)+(b-1)M(a,b-1,z)=0$$

13.4.4

$$bM(a,b,z)-bM(a-1,b,z)-zM(a,b+1,z)=0$$

13.4.5

$$b(a+z)M(a,b,z)+z(a-b)M(a,b+1,z)-abM(a+1,b,z)=0$$