If $\varphi(z, \lambda)$ can be written in the form

10.4.109 $\varphi(z,\lambda)=\lambda^2 p(z)+q(z,\lambda)$

where $q(z, \lambda)$ is bounded in a region R of the z-plane, then the zeros of $p(z)$ in R are said to be turning points of the equation **10.4.108.**

The Special Case $w''+[\lambda^2 z+q(z,\lambda)]w=0$

Let $\lambda=|\lambda|e^{i\omega}$ vary over a sectorial domain S: $|\lambda|\geq\lambda_0(>0)$, $\omega_1\leq\omega\leq\omega_2$, and suppose that $q(z, \lambda)$ is continuous in z for $|z|<r$ and λ in S, and $q(z, \lambda)$ $\sim\sum_0^\infty q_n(z)\lambda^{-n}$ as $\lambda\to\infty$ in S.

Formal Series Solution

10.4.110

$$w(z)=u(z)\sum_0^\infty \varphi_n(z)\lambda^{-n}+\lambda^{-1}u'(z)\sum_0^\infty \psi_n(z)\lambda^{-n}$$

$$u''+\lambda^2 zu=0$$

$$\varphi_0(z)=c_0,\qquad \psi_0(z)=z^{-\frac{1}{2}}c_1,\qquad c_0,\ c_1 \text{ constants}$$

$$\varphi_{n+1}(z)=-\frac{1}{2}\psi_n'(z)-\frac{1}{2}\int_0^z\sum_0^n q_{n-k}(t)\psi_k(t)dt$$

$$\psi_n(z)=\frac{1}{2}z^{-\frac{1}{2}}\int_0^z t^{-\frac{1}{2}}\left[\varphi_n''(t)+\sum_0^n q_{n-k}(t)\varphi_k(t)\right]dt$$

$$(n=0, 1, 2, \ldots)$$

Uniform Asymptotic Expansions of Solutions

For z real, i.e. for the equation

10.4.111 $y''+[\lambda^2 x+q(x,\lambda)]y=0$

where x varies in a bounded interval $a\leq x\leq b$ that includes the origin and where, for each fixed λ in S, $q(x, \lambda)$ is continuous in x for $a\leq x\leq b$, the following asymptotic representations hold.

(i) If λ is real and positive, there are solutions $y_0(x)$, $y_1(x)$ such that, uniformly in x on $a\leq x\leq 0$,

10.4.112

$$y_0(x)=\text{Ai}\,(-\lambda^{2/3}x)[1+O(\lambda^{-1})]\qquad(\lambda\to\infty)$$

$$y_1(x)=\text{Bi}\,(-\lambda^{2/3}x)[1+O(\lambda^{-1})]$$

and, uniformly in x on $0\leq x\leq b$

10.4.113

$$y_0(x)=\text{Ai}\,(-\lambda^{2/3}x)[1+O(\lambda^{-1})]+\text{Bi}\,(-\lambda^{2/3}x)O(\lambda^{-1}),$$

$$y_1(x)=\text{Bi}\,(-\lambda^{2/3}x)[1+O(\lambda^{-1})]+\text{Ai}\,(-\lambda^{2/3}x)O(\lambda^{-1})$$

$$(\lambda\to\infty)$$

(ii) If $\mathscr{R}\lambda\geq 0$, $\mathscr{I}\lambda\neq 0$, there are solutions $y_0(x)$, $y_1(x)$ such that, uniformly in x on $a\leq x\leq b$,

10.4.114

$$y_0(x)=\text{Ai}\,(-\lambda^{2/3}x)[1+O(\lambda^{-1})]$$

$$y_1(x)=\text{Bi}\,(-\lambda^{2/3}x)[1+O(\lambda^{-1})]\qquad(|\lambda|\to\infty)$$

For further representations and details, we refer to [10.4].

When z is complex (bounded or unbounded), conditions under which the formal series **10.4.110** yields a uniform asymptotic expansion of a solution are given in [10.12] if $q(z, \lambda)$ is independent of λ and $|\lambda|\to\infty$ with fixed ω, and in [10.14] if λ lies in any region of the complex plane. Further references are [10.2; 10.9; 10.10].

The General Case $w''+[\lambda^2 p(z)+q(z,\lambda)]w=0$

Let $\lambda=|\lambda|e^{i\omega}$ where $|\lambda|\geq\lambda_0(>0)$ and $-\pi\leq\omega\leq\pi$; suppose that $p(z)$ is analytic in a region R and has a zero $z=z_0$ in R, and that, for fixed λ, $q(z, \lambda)$ is analytic in z for z in R. The transformation $\xi=\xi(z)$, $v=[p(z)/\xi]^{1/4}w(z)$, where ξ is defined as the (unique) solution of the equation

10.4.115 $\xi\left(\dfrac{d\xi}{dz}\right)^2=p(z),$

yields the special case

10.4.116 $\dfrac{d^2v}{d\xi^2}+[\lambda^2\xi+f(\xi,\lambda)]v=0,$ *

$$f(\xi,\lambda)=\left(\frac{d\xi}{dz}\right)^{-2}q(z,\lambda)-\left(\frac{d\xi}{dz}\right)^{-\frac{1}{2}}\frac{d^2}{d\xi^2}\left[\left(\frac{d\xi}{dz}\right)^{\frac{1}{2}}\right].$$

Example:

Consider the equation

10.4.117 $y''+[\lambda^2-(\lambda^2-\frac{1}{4})\,x^{-2}]y=0$

for which the points $x=0$, ∞ are singular points and $x=1$ is a turning point. It has the functions $x^{\frac{1}{2}}J_\lambda(\lambda x)$, $x^{\frac{1}{2}}Y_\lambda(\lambda x)$ as particular solutions (see **9.1.49**).

The equation **10.4.115** becomes

$$\xi\left(\frac{d\xi}{dx}\right)^2=\frac{x^2-1}{x^2}$$

whence

$$\tfrac{2}{3}(-\xi)^{3/2}=-\sqrt{1-x^2}+\ln x^{-1}(1+\sqrt{1-x^2})\qquad(0<x\leq 1)$$

$$\tfrac{2}{3}\xi^{3/2}=\sqrt{x^2-1}-\arccos x^{-1}\qquad(1\leq x<\infty).$$

Thus

10.4.118 $v(\xi)=\left(\dfrac{x^2-1}{x^2\xi}\right)^{1/4}y(x)$

*See page II.