### The Functions Gi(z), Hi(z)

**10.4.42**

$$\mathrm{Gi}\,(z)=\pi^{-1}\int_0^\infty \sin\left(\frac{1}{3}t^3+zt\right)dt$$
$$=\frac{1}{3}\mathrm{Bi}\,(z)+\int_0^z[\mathrm{Ai}\,(z)\,\mathrm{Bi}\,(t)-\mathrm{Ai}\,(t)\,\mathrm{Bi}\,(z)]dt$$

**10.4.43**

$$\mathrm{Gi}'\,(z)=\frac{1}{3}\mathrm{Bi}'\,(z)+\int_0^z[\mathrm{Ai}'(z)\,\mathrm{Bi}\,(t)-\mathrm{Ai}\,(t)\,\mathrm{Bi}'(z)]dt$$

**10.4.44**

$$\mathrm{Hi}(z)=\pi^{-1}\int_0^\infty \exp\left(-\frac{1}{3}t^3+zt\right)dt$$
$$=\frac{2}{3}\mathrm{Bi}\,(z)+\int_0^z[\mathrm{Ai}\,(t)\,\mathrm{Bi}\,(z)-\mathrm{Ai}(z)\,\mathrm{Bi}\,(t)]dt$$

**10.4.45**

$$\mathrm{Hi}'(z)=\frac{2}{3}\mathrm{Bi}'\,(z)+\int_0^z[\mathrm{Ai}\,(t)\,\mathrm{Bi}'(z)-\mathrm{Ai}'(z)\,\mathrm{Bi}\,(t)]dt$$

**10.4.46** $$\mathrm{Gi}\,(z)+\mathrm{Hi}\,(z)=\mathrm{Bi}\,(z)$$

### Representations of $\int_0^z \mathrm{Ai}(\pm t)dt$, $\int_0^z \mathrm{Bi}(\pm t)dt$ by Gi $(\pm z)$, Hi $(\pm z)$

**10.4.47**

$$\int_0^z \mathrm{Ai}\,(t)dt=\frac{1}{3}+\pi[\mathrm{Ai}'\,(z)\mathrm{Gi}(z)-\mathrm{Ai}\,(z)\mathrm{Gi}'(z)]$$

**10.4.48**

$$=-\frac{2}{3}-\pi[\mathrm{Ai}'\,(z)\,\mathrm{Hi}\,(z)-\mathrm{Ai}\,(z)\,\mathrm{Hi}'\,(z)]$$

**10.4.49**

$$\int_0^z \mathrm{Ai}\,(-t)dt=-\frac{1}{3}-\pi[\mathrm{Ai}'\,(-z)\,\mathrm{Gi}\,(-z)-\mathrm{Ai}\,(-z)\,\mathrm{Gi}'\,(-z)]$$

**10.4.50**

$$=\frac{2}{3}+\pi[\mathrm{Ai}'\,(-z)\,\mathrm{Hi}\,(-z)-\mathrm{Ai}\,(-z)\mathrm{Hi}'(-z)]$$

**10.4.51**

$$\int_0^z \mathrm{Bi}\,(t)dt=\pi[\mathrm{Bi}'\,(z)\,\mathrm{Gi}\,(z)-\mathrm{Bi}\,(z)\,\mathrm{Gi}'\,(z)]$$

**10.4.52** $$=-\pi[\mathrm{Bi}'\,(z)\,\mathrm{Hi}\,(z)-\mathrm{Bi}\,(z)\,\mathrm{Hi}'(z)]$$

**10.4.53**

$$\int_0^z \mathrm{Bi}\,(-t)dt=-\pi[\mathrm{Bi}'\,(-z)\,\mathrm{Gi}\,(-z)-\mathrm{Bi}\,(-z)\,\mathrm{Gi}'\,(-z)]$$

**10.4.54** $$=\pi[\mathrm{Bi}'\,(-z)\,\mathrm{Hi}\,(-z)-\mathrm{Bi}\,(-z)\,\mathrm{Hi}'\,(-z)]$$

### Differential Equations for Gi (z), Hi (z)

**10.4.55** $$w''-zw=-\pi^{-1}$$

$$w(0)=\frac{1}{3}\mathrm{Bi}\,(0)=\frac{1}{\sqrt{3}}\mathrm{Ai}\,(0)=.20497\ 55424\ 78$$

$$w'(0)=\frac{1}{3}\mathrm{Bi}'\,(0)=-\frac{1}{\sqrt{3}}\mathrm{Ai}'\,(0)=.14942\ 94524\ 49$$

$$w(z)=\mathrm{Gi}(z)$$

**10.4.56** $$w''-zw=\pi^{-1}$$

$$w(0)=\frac{2}{3}\mathrm{Bi}\,(0)=\frac{2}{\sqrt{3}}\mathrm{Ai}\,(0)=.40995\ 10849\ 56$$

$$w'(0)=\frac{2}{3}\mathrm{Bi}'\,(0)=-\frac{2}{\sqrt{3}}\mathrm{Ai}'\,(0)=.29885\ 89048\ 98$$

$$w(z)=\mathrm{Hi}\,(z)$$

### Differential Equation for Products of Airy Functions

**10.4.57** $$w'''-4zw'-2w=0$$

Linearly independent solutions are $\mathrm{Ai}^2\,(z)$, $\mathrm{Ai}\,(z)\,\mathrm{Bi}\,(z)$, $\mathrm{Bi}^2\,(z)$.

### Wronskian for Products of Airy Functions

**10.4.58** $$W\{\mathrm{Ai}^2\,(z),\mathrm{Ai}\,(z)\,\mathrm{Bi}\,(z),\mathrm{Bi}^2\,(z)\}=2\pi^{-3}$$

### Asymptotic Expansions for |z| Large

$$c_0=1,\ c_k=\frac{\Gamma(3k+\frac{1}{2})}{54^k k!\,\Gamma(k+\frac{1}{2})}=\frac{(2k+1)(2k+3)\ldots(6k-1)}{216^k k!},$$

$$d_0=1,\ d_k=-\frac{6k+1}{6k-1}c_k \qquad (k=1,2,3,\ldots)$$

$$\zeta=\frac{2}{3}z^{3/2}$$

**10.4.59**

$$\mathrm{Ai}\,(z)\sim\frac{1}{2}\pi^{-1/2}z^{-1/4}e^{-\zeta}\sum_0^\infty(-1)^k c_k\zeta^{-k} \qquad (|\arg z|<\pi)$$

**10.4.60**

$$\mathrm{Ai}\,(-z)\sim\pi^{-1/2}z^{-1/4}\left[\sin\left(\zeta+\frac{\pi}{4}\right)\sum_0^\infty(-1)^k c_{2k}\zeta^{-2k}-\cos\left(\zeta+\frac{\pi}{4}\right)\sum_0^\infty(-1)^k c_{2k+1}\zeta^{-2k-1}\right]$$

$$(|\arg z|<\tfrac{2}{3}\pi)$$

**10.4.61**

$$\mathrm{Ai}'\,(z)\sim-\tfrac{1}{2}\pi^{-1/2}z^{1/4}e^{-\zeta}\sum_0^\infty(-1)^k d_k\zeta^{-k}$$

$$(|\arg z|<\pi)$$