Spherical Bessel Functions of the Second and Third Kind

10.1.15

$$y_n(z)=(-1)^{n+1}j_{-n-1}(z) \qquad (n=0, \pm 1, \pm 2, \ldots)$$

10.1.16

$$h_n^{(1)}(z)=i^{-n-1}z^{-1}e^{iz}\sum_0^n (n+\tfrac{1}{2}, k)(-2iz)^{-k}$$

10.1.17

$$h_n^{(2)}(z)=i^{n+1}z^{-1}e^{-iz}\sum_0^n (n+\tfrac{1}{2}, k)\,(2iz)^{-k} \qquad *$$

10.1.18

$$h_{-n-1}^{(1)}(z)=i(-1)^n h_n^{(1)}(z)$$

$$h_{-n-1}^{(2)}(z)=-i(-1)^n h_n^{(2)}(z) \qquad (n=0, 1, 2, \ldots)$$

Elementary Properties

Recurrence Relations

$f_n(z): j_n(z), y_n(z), h_n^{(1)}(z), h_n^{(2)}(z)$

$(n=0, \pm 1, \pm 2, \ldots)$

10.1.19 $$f_{n-1}(z)+f_{n+1}(z)=(2n+1)z^{-1}f_n(z)$$

10.1.20 $$nf_{n-1}(z)-(n+1)f_{n+1}(z)=(2n+1)\frac{d}{dz}f_n(z)$$

10.1.21 $$\frac{n+1}{z}f_n(z)+\frac{d}{dz}f_n(z)=f_{n-1}(z)$$

(See **10.1.23.**)

10.1.22 $$\frac{n}{z}f_n(z)-\frac{d}{dz}f_n(z)=f_{n+1}(z)$$

(See **10.1.24.**)

Differentiation Formulas

$f_n(z): j_n(z), y_n(z), h_n^{(1)}(z), h_n^{(2)}(z)$

$(n=0, \pm 1, \pm 2, \ldots)$

10.1.23 $$\left(\frac{1}{z}\frac{d}{dz}\right)^m[z^{n+1}f_n(z)]=z^{n-m+1}f_{n-m}(z)$$

10.1.24 $$\left(\frac{1}{z}\frac{d}{dz}\right)^m[z^{-n}f_n(z)]=(-1)^m z^{-n-m}f_{n+m}(z)$$

$(m=1, 2, 3, \ldots)$

Rayleigh's Formulas

10.1.25

$$j_n(z)=z^n\left(-\frac{1}{z}\frac{d}{dz}\right)^n\frac{\sin z}{z}$$

10.1.26

$$y_n(z)=-z^n\left(-\frac{1}{z}\frac{d}{dz}\right)^n\frac{\cos z}{z} \qquad (n=0, 1, 2, \ldots)$$

*See page II.

Modulus and Phase

$$j_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,M_{n+\frac{1}{2}}(z)\cos\theta_{n+\frac{1}{2}}(z),$$

$$y_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,M_{n+\frac{1}{2}}(z)\sin\theta_{n+\frac{1}{2}}(z)$$

(See **9.2.17.**)

10.1.27

$$(\tfrac{1}{2}\pi/z)\,M_{n+\frac{1}{2}}^2(z)=\frac{1}{z^2}\sum_0^n\frac{(2n-k)!(2n-2k)!}{k![(n-k)!]^2}(2z)^{2k-2n}$$

(See **9.2.28.**)

10.1.28 $$(\tfrac{1}{2}\pi/z)M_{1/2}^2(z)=j_0^2(z)+y_0^2(z)=z^{-2}$$

10.1.29

$$(\tfrac{1}{2}\pi/z)M_{3/2}^2(z)=j_1^2(z)+y_1^2(z)=z^{-2}+z^{-4}$$

10.1.30

$$(\tfrac{1}{2}\pi/z)M_{5/2}^2(z)=j_2^2(z)+y_2^2(z)=z^{-2}+3z^{-4}+9z^{-6}$$

Cross Products

10.1.31 $$j_n(z)y_{n-1}(z)-j_{n-1}(z)y_n(z)=z^{-2}$$

10.1.32

$$j_{n+1}(z)y_{n-1}(z)-j_{n-1}(z)y_{n+1}(z)=(2n+1)z^{-3}$$

10.1.33

$$j_0(z)j_n(z)+y_0(z)y_n(z)$$

$$=z^{-2}\sum_0^{[\frac{1}{2}n]}(-1)^k 2^{n-2k}\left(k+\tfrac{1}{2}\right)_{n-2k}\binom{n-k}{k}z^{2k-n}$$

$(n=0, 1, 2, \ldots)$

Analytic Continuation

10.1.34 $$j_n(ze^{m\pi i})=e^{mn\pi i}j_n(z)$$

10.1.35 $$y_n(ze^{m\pi i})=(-1)^m e^{mn\pi i}y_n(z)$$

10.1.36 $$h_n^{(1)}(ze^{(2m+1)\pi i})=(-1)^n h_n^{(2)}(z)$$

10.1.37 $$h_n^{(2)}(ze^{(2m+1)\pi i})=(-1)^n h_n^{(1)}(z)$$

10.1.38 $$h_n^{(l)}(ze^{2m\pi i})=h_n^{(l)}(z)$$

$(l=1, 2; m, n=0, 1, 2, \ldots)$

Generating Functions

10.1.39

$$\frac{1}{z}\sin\sqrt{z^2+2zt}=\sum_0^\infty\frac{(-t)^n}{n!}y_{n-1}(z) \qquad (2|t|<|z|)$$

10.1.40 $$\frac{1}{z}\cos\sqrt{z^2-2zt}=\sum_0^\infty\frac{t^n}{n!}j_{n-1}(z)$$