Asymptotic Expansions of Modulus and Phase

When ν is fixed, x is large and $\mu=4\nu^2$

9.10.21

$$M_\nu=\frac{e^{x/\sqrt{2}}}{\sqrt{2\pi x}}\left\{1-\frac{\mu-1}{8\sqrt{2}}\frac{1}{x}+\frac{(\mu-1)^2}{256}\frac{1}{x^2}-\frac{(\mu-1)(\mu^2+14\mu-399)}{6144\sqrt{2}}\frac{1}{x^3}+O\left(\frac{1}{x^4}\right)\right\}$$

9.10.22

$$\ln M_\nu=\frac{x}{\sqrt{2}}-\tfrac{1}{2}\ln(2\pi x)-\frac{\mu-1}{8\sqrt{2}}\frac{1}{x}-\frac{(\mu-1)(\mu-25)}{384\sqrt{2}}\frac{1}{x^3}-\frac{(\mu-1)(\mu-13)}{128}\frac{1}{x^4}+O\left(\frac{1}{x^5}\right)$$

9.10.23

$$\theta_\nu=\frac{x}{\sqrt{2}}+\left(\frac{1}{2}\nu-\frac{1}{8}\right)\pi+\frac{\mu-1}{8\sqrt{2}}\frac{1}{x}+\frac{\mu-1}{16}\frac{1}{x^2}-\frac{(\mu-1)(\mu-25)}{384\sqrt{2}}\frac{1}{x^3}+O\left(\frac{1}{x^5}\right)$$

9.10.24

$$N_\nu=\sqrt{\frac{\pi}{2x}}\,e^{-x/\sqrt{2}}\left\{1+\frac{\mu-1}{8\sqrt{2}}\frac{1}{x}+\frac{(\mu-1)^2}{256}\frac{1}{x^2}+\frac{(\mu-1)(\mu^2+14\mu-399)}{6144\sqrt{2}}\frac{1}{x^3}+O\left(\frac{1}{x^4}\right)\right\}$$

9.10.25

$$\ln N_\nu=-\frac{x}{\sqrt{2}}+\tfrac{1}{2}\ln\left(\frac{\pi}{2x}\right)+\frac{\mu-1}{8\sqrt{2}}\frac{1}{x}+\frac{(\mu-1)(\mu-25)}{384\sqrt{2}}\frac{1}{x^3}-\frac{(\mu-1)(\mu-13)}{128}\frac{1}{x^4}+O\left(\frac{1}{x^5}\right)$$

9.10.26

$$\phi_\nu=-\frac{x}{\sqrt{2}}-\left(\frac{1}{2}\nu+\frac{1}{8}\right)\pi-\frac{\mu-1}{8\sqrt{2}}\frac{1}{x}+\frac{\mu-1}{16}\frac{1}{x^2}+\frac{(\mu-1)(\mu-25)}{384\sqrt{2}}\frac{1}{x^3}+O\left(\frac{1}{x^5}\right)$$

Asymptotic Expansions of Cross-Products

If x is large

9.10.27

$$\mathrm{ber}^2 x+\mathrm{bei}^2 x\sim\frac{e^{x\sqrt{2}}}{2\pi x}\left(1+\frac{1}{4\sqrt{2}}\frac{1}{x}+\frac{1}{64}\frac{1}{x^2}-\frac{33}{256\sqrt{2}}\frac{1}{x^3}-\frac{1797}{8192}\frac{1}{x^4}+\ldots\right)$$

9.10.28

$$\mathrm{ber}\,x\,\mathrm{bei}'\,x-\mathrm{ber}'\,x\,\mathrm{bei}\,x\sim\frac{e^{x\sqrt{2}}}{2\pi x}\left(\frac{1}{\sqrt{2}}+\frac{1}{8}\frac{1}{x}+\frac{9}{64\sqrt{2}}\frac{1}{x^2}+\frac{39}{512}\frac{1}{x^3}+\frac{75}{8192\sqrt{2}}\frac{1}{x^4}+\ldots\right)$$

9.10.29

$$\mathrm{ber}\,x\,\mathrm{ber}'\,x+\mathrm{bei}\,x\,\mathrm{bei}'\,x\sim\frac{e^{x\sqrt{2}}}{2\pi x}\left(\frac{1}{\sqrt{2}}-\frac{3}{8}\frac{1}{x}-\frac{15}{64\sqrt{2}}\frac{1}{x^2}-\frac{45}{512}\frac{1}{x^3}+\frac{315}{8192\sqrt{2}}\frac{1}{x^4}+\ldots\right)$$

9.10.30

$$\mathrm{ber}'^2 x+\mathrm{bei}'^2 x\sim\frac{e^{x\sqrt{2}}}{2\pi x}\left(1-\frac{3}{4\sqrt{2}}\frac{1}{x}+\frac{9}{64}\frac{1}{x^2}+\frac{75}{256\sqrt{2}}\frac{1}{x^3}+\frac{2475}{8192}\frac{1}{x^4}+\ldots\right)$$

9.10.31

$$\mathrm{ker}^2 x+\mathrm{kei}^2 x\sim\frac{\pi}{2x}e^{-x\sqrt{2}}\left(1-\frac{1}{4\sqrt{2}}\frac{1}{x}+\frac{1}{64}\frac{1}{x^2}+\frac{33}{256\sqrt{2}}\frac{1}{x^3}-\frac{1797}{8192}\frac{1}{x^4}+\ldots\right)$$

9.10.32

$$\mathrm{ker}\,x\,\mathrm{kei}'\,x-\mathrm{ker}'\,x\,\mathrm{kei}\,x\sim-\frac{\pi}{2x}e^{-x\sqrt{2}}\left(\frac{1}{\sqrt{2}}-\frac{1}{8}\frac{1}{x}+\frac{9}{64\sqrt{2}}\frac{1}{x^2}-\frac{39}{512}\frac{1}{x^3}+\frac{75}{8192\sqrt{2}}\frac{1}{x^4}+\ldots\right)$$

9.10.33

$$\mathrm{ker}\,x\,\mathrm{ker}'\,x+\mathrm{kei}\,x\,\mathrm{kei}'\,x\sim-\frac{\pi}{2x}e^{-x\sqrt{2}}\left(\frac{1}{\sqrt{2}}+\frac{3}{8}\frac{1}{x}-\frac{15}{64\sqrt{2}}\frac{1}{x^2}+\frac{45}{512}\frac{1}{x^3}+\frac{315}{8192\sqrt{2}}\frac{1}{x^4}+\ldots\right)$$

9.10.34

$$\mathrm{ker}'^2 x+\mathrm{kei}'^2 x\sim\frac{\pi}{2x}e^{-x\sqrt{2}}\left(1+\frac{3}{4\sqrt{2}}\frac{1}{x}+\frac{9}{64}\frac{1}{x^2}-\frac{75}{256\sqrt{2}}\frac{1}{x^3}+\frac{2475}{8192}\frac{1}{x^4}+\cdots\right)$$

Asymptotic Expansions of Large Zeros

Let

9.10.35

$$f(\delta)=\frac{\mu-1}{16\delta}+\frac{\mu-1}{32\delta^2}+\frac{(\mu-1)(5\mu+19)}{1536\delta^3}+\frac{3(\mu-1)^2}{512\delta^4}+\cdots$$

where $\mu=4\nu^2$. Then if s is a large positive integer

9.10.36

Zeros of $\mathrm{ber}_\nu\,x\sim\sqrt{2}\{\delta-f(\delta)\}$, $\quad\delta=(s-\tfrac{1}{2}\nu-\tfrac{3}{8})\pi$

Zeros of $\mathrm{bei}_\nu\,x\sim\sqrt{2}\{\delta-f(\delta)\}$, $\quad\delta=(s-\tfrac{1}{2}\nu+\tfrac{1}{8})\pi$

Zeros of $\mathrm{ker}_\nu\,x\sim\sqrt{2}\{\delta+f(-\delta)\}$, $\quad\delta=(s-\tfrac{1}{2}\nu-\tfrac{5}{8})\pi$

Zeros of $\mathrm{kei}_\nu\,x\sim\sqrt{2}\{\delta+f(-\delta)\}$, $\quad\delta=(s-\tfrac{1}{2}\nu-\tfrac{1}{8})\pi$