$$\operatorname{kei}_n x=-\tfrac{1}{2}(\tfrac{1}{2}x)^{-n}\sum_{k=0}^{n-1}\sin\{(\tfrac{3}{4}n+\tfrac{1}{2}k)\pi\}\times\frac{(n-k-1)!}{k!}(\tfrac{1}{4}x^2)^k-\ln(\tfrac{1}{2}x)\operatorname{bei}_n x-\tfrac{1}{4}\pi\operatorname{ber}_n x+\tfrac{1}{2}(\tfrac{1}{2}x)^n\sum_{k=0}^{\infty}\sin\{(\tfrac{3}{4}n+\tfrac{1}{2}k)\pi\}\times\frac{\{\psi(k+1)+\psi(n+k+1)\}}{k!(n+k)!}(\tfrac{1}{4}x^2)^k$$

where $\psi(n)$ is given by **6.3.2.**

9.9.12

$$\ker x=-\ln(\tfrac{1}{2}x)\operatorname{ber} x+\tfrac{1}{4}\pi\operatorname{bei} x+\sum_{k=0}^{\infty}(-)^k\frac{\psi(2k+1)}{\{(2k)!\}^2}(\tfrac{1}{4}x^2)^{2k}$$

$$\operatorname{kei} x=-\ln(\tfrac{1}{2}x)\operatorname{bei} x-\tfrac{1}{4}\pi\operatorname{ber} x+\sum_{k=0}^{\infty}(-)^k\frac{\psi(2k+2)}{\{(2k+1)!\}^2}(\tfrac{1}{4}x^2)^{2k+1}$$

Functions of Negative Argument

In general Kelvin functions have a branch point at $x=0$ and individual functions with arguments $xe^{\pm\pi i}$ are complex. The branch point is absent however in the case of ber_ν and bei_ν when ν is an integer, and

9.9.13

$$\operatorname{ber}_n(-x)=(-)^n\operatorname{ber}_n x,\qquad \operatorname{bei}_n(-x)=(-)^n\operatorname{bei}_n x$$

Recurrence Relations

9.9.14

$$f_{\nu+1}+f_{\nu-1}=-\frac{\nu\sqrt{2}}{x}(f_\nu-g_\nu)$$

$$f'_\nu=\frac{1}{2\sqrt{2}}(f_{\nu+1}+g_{\nu+1}-f_{\nu-1}-g_{\nu-1})$$

$$f'_\nu-\frac{\nu}{x}f_\nu=\frac{1}{\sqrt{2}}(f_{\nu+1}+g_{\nu+1})$$

$$f'_\nu+\frac{\nu}{x}f_\nu=-\frac{1}{\sqrt{2}}(f_{\nu-1}+g_{\nu-1})$$

where

9.9.15

$$\left.\begin{aligned}f_\nu&=\operatorname{ber}_\nu x\\ g_\nu&=\operatorname{bei}_\nu x\end{aligned}\right\}\qquad\left.\begin{aligned}f_\nu&=\operatorname{bei}_\nu x\\ g_\nu&=-\operatorname{ber}_\nu x\end{aligned}\right\}$$

$$\left.\begin{aligned}f_\nu&=\ker_\nu x\\ g_\nu&=\operatorname{kei}_\nu x\end{aligned}\right\}\qquad\left.\begin{aligned}f_\nu&=\operatorname{kei}_\nu x\\ g_\nu&=-\ker_\nu x\end{aligned}\right\}$$

9.9.16

$$\sqrt{2}\operatorname{ber}' x=\operatorname{ber}_1 x+\operatorname{bei}_1 x$$

$$\sqrt{2}\operatorname{bei}' x=-\operatorname{ber}_1 x+\operatorname{bei}_1 x$$

9.9.17

$$\sqrt{2}\ker' x=\ker_1 x+\operatorname{kei}_1 x$$

$$\sqrt{2}\operatorname{kei}' x=-\ker_1 x+\operatorname{kei}_1 x$$

Recurrence Relations for Cross-Products

If

9.9.18

$$p_\nu=\operatorname{ber}_\nu^2 x+\operatorname{bei}_\nu^2 x$$

$$q_\nu=\operatorname{ber}_\nu x\operatorname{bei}'_\nu x-\operatorname{ber}'_\nu x\operatorname{bei}_\nu x$$

$$r_\nu=\operatorname{ber}_\nu x\operatorname{ber}'_\nu x+\operatorname{bei}_\nu x\operatorname{bei}'_\nu x$$

$$s_\nu=\operatorname{ber}'^2_\nu x+\operatorname{bei}'^2_\nu x$$

then

9.9.19

$$p_{\nu+1}=p_{\nu-1}-\frac{4\nu}{x}r_\nu$$

$$q_{\nu+1}=-\frac{\nu}{x}p_\nu+r_\nu=-q_{\nu-1}+2r_\nu$$

$$r_{\nu+1}=-\frac{(\nu+1)}{x}p_{\nu+1}+q_\nu$$

$$s_\nu=\frac{1}{2}p_{\nu+1}+\frac{1}{2}p_{\nu-1}-\frac{\nu^2}{x^2}p_\nu$$

and

9.9.20 $$p_\nu s_\nu=r_\nu^2+q_\nu^2$$

The same relations hold with ber, bei replaced throughout by ker, kei, respectively.

Indefinite Integrals

In the following f_ν, g_ν are any one of the pairs given by equations **9.9.15** and f^*_ν, g^*_ν are either the same pair or any other pair.

9.9.21

$$\int x^{1+\nu}f_\nu dx=-\frac{x^{1+\nu}}{\sqrt{2}}(f_{\nu+1}-g_{\nu+1})=-x^{1+\nu}\left(\frac{\nu}{x}g_\nu-g'_\nu\right)$$

9.9.22

$$\int x^{1-\nu}f_\nu dx=\frac{x^{1-\nu}}{\sqrt{2}}(f_{\nu-1}-g_{\nu-1})=x^{1-\nu}\left(\frac{\nu}{x}g_\nu+g'_\nu\right)$$

9.9.23

$$\int x(f_\nu g^*_\nu-g_\nu f^*_\nu)dx=\frac{x}{2\sqrt{2}}\{f^*_\nu(f_{\nu+1}+g_{\nu+1})-g^*_\nu(f_{\nu+1}-g_{\nu+1})-f_\nu(f^*_{\nu+1}+g^*_{\nu+1})+g_\nu(f^*_{\nu+1}-g^*_{\nu+1})\}=\frac{1}{2}x(f'_\nu f^*_\nu-f_\nu f^{*\prime}_\nu+g'_\nu g^*_\nu-g_\nu g^{*\prime}_\nu)$$