Integral Representations

9.6.16
$$I_0(z)=\frac{1}{\pi}\int_0^\pi e^{\pm z\cos\theta}d\theta=\frac{1}{\pi}\int_0^\pi \cosh(z\cos\theta)d\theta$$

9.6.17 $$K_0(z)=-\frac{1}{\pi}\int_0^\pi e^{\pm z\cos\theta}\{\gamma+\ln(2z\sin^2\theta)\}d\theta$$

9.6.18
$$I_\nu(z)=\frac{(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\int_0^\pi e^{\pm z\cos\theta}\sin^{2\nu}\theta\,d\theta$$
$$=\frac{(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}}e^{\pm zt}dt \qquad (\mathscr{R}\nu>-\tfrac{1}{2})$$

9.6.19 $$I_n(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)d\theta$$

9.6.20
$$I_\nu(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta)d\theta$$
$$-\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt \qquad (|\arg z|<\tfrac{1}{2}\pi)$$

9.6.21
$$K_0(x)=\int_0^\infty \cos(x\sinh t)dt=\int_0^\infty \frac{\cos(xt)}{\sqrt{t^2+1}}dt \qquad (x>0)$$

9.6.22
$$K_\nu(x)=\sec(\tfrac{1}{2}\nu\pi)\int_0^\infty \cos(x\sinh t)\cosh(\nu t)dt$$
$$=\csc(\tfrac{1}{2}\nu\pi)\int_0^\infty \sin(x\sinh t)\sinh(\nu t)dt \qquad (|\mathscr{R}\nu|<1,\ x>0)$$

9.6.23
$$K_\nu(z)=\frac{\pi^{\frac{1}{2}}(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty e^{-z\cosh t}\sinh^{2\nu}t\,dt$$
$$=\frac{\pi^{\frac{1}{2}}(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})}\int_1^\infty e^{-zt}(t^2-1)^{\nu-\frac{1}{2}}\,dt \qquad (\mathscr{R}\nu>-\tfrac{1}{2},\ |\arg z|<\tfrac{1}{2}\pi)$$

9.6.24 $$K_\nu(z)=\int_0^\infty e^{-z\cosh t}\cosh(\nu t)dt \qquad (|\arg z|<\tfrac{1}{2}\pi)$$

9.6.25
$$K_\nu(xz)=\frac{\Gamma(\nu+\frac{1}{2})(2z)^\nu}{\pi^{\frac{1}{2}}x^\nu}\int_0^\infty \frac{\cos(xt)dt}{(t^2+z^2)^{\nu+\frac{1}{2}}}$$
$$(\mathscr{R}\nu>-\tfrac{1}{2},\ x>0,\ |\arg z|<\tfrac{1}{2}\pi)^*$$

Recurrence Relations

9.6.26
$$\mathscr{Z}_{\nu-1}(z)-\mathscr{Z}_{\nu+1}(z)=\frac{2\nu}{z}\mathscr{Z}_\nu(z)$$
$$\mathscr{Z}'_\nu(z)=\mathscr{Z}_{\nu-1}(z)-\frac{\nu}{z}\mathscr{Z}_\nu(z)$$
$$\mathscr{Z}_{\nu-1}(z)+\mathscr{Z}_{\nu+1}(z)=2\mathscr{Z}'_\nu(z)$$
$$\mathscr{Z}'_\nu(z)=\mathscr{Z}_{\nu+1}(z)+\frac{\nu}{z}\mathscr{Z}_\nu(z)$$

$\mathscr{Z}_\nu$ denotes I_ν, $e^{\nu\pi i}K_\nu$ or any linear combination of these functions, the coefficients in which are independent of z and ν.

9.6.27 $$I'_0(z)=I_1(z), \qquad K'_0(z)=-K_1(z)$$

Formulas for Derivatives

9.6.28
$$\left(\frac{1}{z}\frac{d}{dz}\right)^k\{z^\nu\mathscr{Z}_\nu(z)\}=z^{\nu-k}\mathscr{Z}_{\nu-k}(z)$$
$$\left(\frac{1}{z}\frac{d}{dz}\right)^k\{z^{-\nu}\mathscr{Z}_\nu(z)\}=z^{-\nu-k}\mathscr{Z}_{\nu+k}(z) \qquad (k=0,1,2,\ldots)$$

9.6.29
$$\mathscr{Z}_\nu^{(k)}(z)=\frac{1}{2^k}\left\{\mathscr{Z}_{\nu-k}(z)+\binom{k}{1}\mathscr{Z}_{\nu-k+2}(z)+\binom{k}{2}\mathscr{Z}_{\nu-k+4}(z)+\ldots+\mathscr{Z}_{\nu+k}(z)\right\}$$
$$(k=0,1,2,\ldots)$$

Analytic Continuation

9.6.30 $$I_\nu(ze^{m\pi i})=e^{m\nu\pi i}I_\nu(z) \qquad (m \text{ an integer})$$

9.6.31
$$K_\nu(ze^{m\pi i})=e^{-m\nu\pi i}K_\nu(z)-\pi i\sin(m\nu\pi)\csc(\nu\pi)I_\nu(z) \qquad (m \text{ an integer})$$

9.6.32 $$I_\nu(\bar{z})=\overline{I_\nu(z)}, \qquad K_\nu(\bar{z})=\overline{K_\nu(z)} \qquad (\nu \text{ real})$$

Generating Function and Associated Series

9.6.33 $$e^{\frac{1}{2}z(t+1/t)}=\sum_{k=-\infty}^{\infty} t^kI_k(z) \qquad (t\neq 0)$$

9.6.34 $$e^{z\cos\theta}=I_0(z)+2\sum_{k=1}^\infty I_k(z)\cos(k\theta)$$

9.6.35
$$e^{z\sin\theta}=I_0(z)+2\sum_{k=0}^\infty(-)^kI_{2k+1}(z)\sin\{(2k+1)\theta\}+2\sum_{k=1}^\infty(-)^kI_{2k}(z)\cos(2k\theta)$$

9.6.36 $$1=I_0(z)-2I_2(z)+2I_4(z)-2I_6(z)+\ldots$$

9.6.37 $$e^z=I_0(z)+2I_1(z)+2I_2(z)+2I_3(z)+\ldots$$

9.6.38 $$e^{-z}=I_0(z)-2I_1(z)+2I_2(z)-2I_3(z)+\ldots$$

9.6.39
$$\cosh z=I_0(z)+2I_2(z)+2I_4(z)+2I_6(z)+\ldots$$

9.6.40 $$\sinh z=2I_1(z)+2I_3(z)+2I_5(z)+\ldots$$

*See page II.