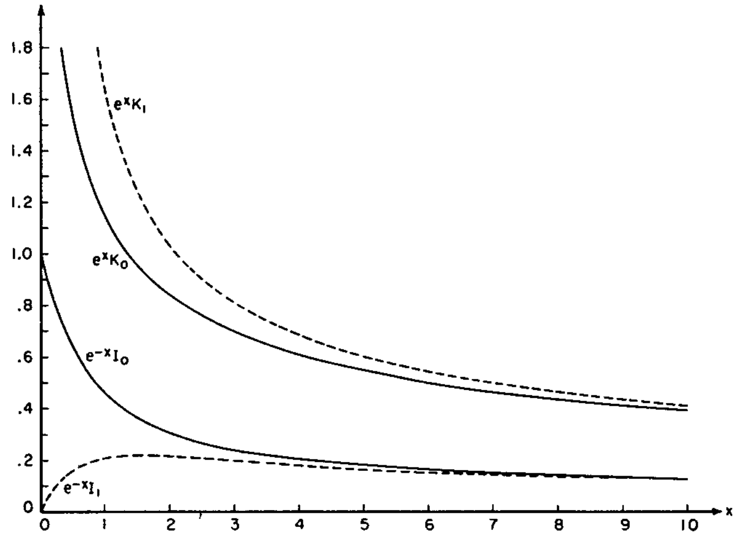


FIGURE 9.8. $e^{-x}I_0(x), e^{-x}I_1(x), e^xK_0(x)$ *and* $e^xK_1(x)$.

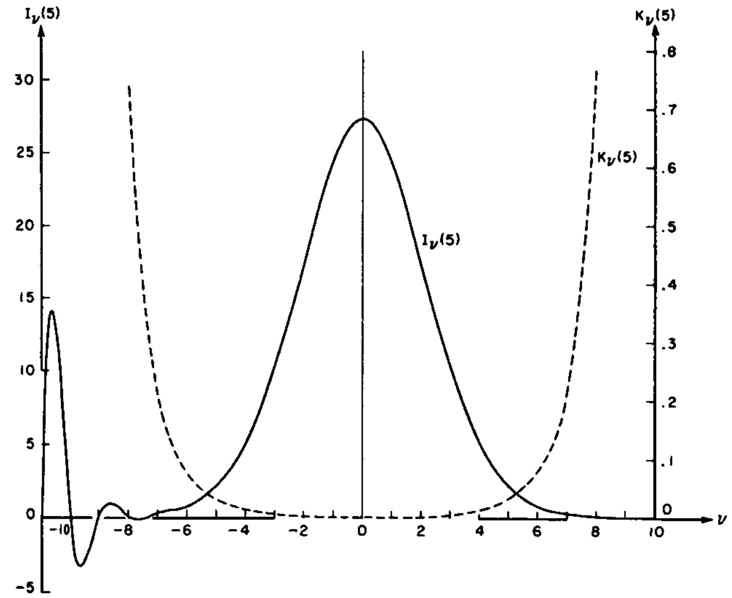


FIGURE 9.9. $I_\nu(5)$ *and* $K_\nu(5)$.

### Relations Between Solutions

**9.6.2**
$$K_\nu(z)=\tfrac12\pi\,\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\nu\pi)}$$

The right of this equation is replaced by its limiting value if $\nu$ is an integer or zero.

**9.6.3**
$$I_\nu(z)=e^{-\frac12\nu\pi i}J_\nu(ze^{\frac12\pi i})\qquad(-\pi<\arg z\le\tfrac12\pi)$$
$$I_\nu(z)=e^{3\nu\pi i/2}J_\nu(ze^{-3\pi i/2})\qquad(\tfrac12\pi<\arg z\le\pi)$$

**9.6.4**
$$K_\nu(z)=\tfrac12\pi i e^{\frac12\nu\pi i}H_\nu^{(1)}(ze^{\frac12\pi i})\qquad(-\pi<\arg z\le\tfrac12\pi)$$
$$K_\nu(z)=-\tfrac12\pi i e^{-\frac12\nu\pi i}H_\nu^{(2)}(ze^{-\frac12\pi i})\quad(-\tfrac12\pi<\arg z\le\pi)$$

**9.6.5**
$$Y_\nu(ze^{\frac12\pi i})=e^{\frac12(\nu+1)\pi i}I_\nu(z)-(2/\pi)e^{-\frac12\nu\pi i}K_\nu(z)\qquad(-\pi<\arg z\le\tfrac12\pi)$$

**9.6.6** $$I_{-n}(z)=I_n(z),\ K_{-\nu}(z)=K_\nu(z)$$

Most of the properties of modified Bessel functions can be deduced immediately from those of ordinary Bessel functions by application of these relations.

### Limiting Forms for Small Arguments

When $\nu$ is fixed and $z\to0$

**9.6.7**
$$I_\nu(z)\sim(\tfrac12z)^\nu/\Gamma(\nu+1)\qquad(\nu\neq-1,-2,\ldots)$$

**9.6.8** $$K_0(z)\sim-\ln z$$

**9.6.9** $$K_\nu(z)\sim\tfrac12\Gamma(\nu)(\tfrac12z)^{-\nu}\qquad(\mathscr{R}\nu>0)$$

### Ascending Series

**9.6.10**
$$I_\nu(z)=(\tfrac12z)^\nu\sum_{k=0}^\infty\frac{(\frac14z^2)^k}{k!\,\Gamma(\nu+k+1)}$$

**9.6.11**
$$K_n(z)=\tfrac12(\tfrac12z)^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(-\tfrac14z^2)^k+(-)^{n+1}\ln(\tfrac12z)I_n(z)$$
$$+(-)^n\tfrac12(\tfrac12z)^n\sum_{k=0}^\infty\{\psi(k+1)+\psi(n+k+1)\}\frac{(\frac14z^2)^k}{k!(n+k)!}$$

where $\psi(n)$ is given by **6.3.2.**

**9.6.12**
$$I_0(z)=1+\frac{\frac14z^2}{(1!)^2}+\frac{(\frac14z^2)^2}{(2!)^2}+\frac{(\frac14z^2)^3}{(3!)^2}+\ldots$$

**9.6.13**
$$K_0(z)=-\{\ln(\tfrac12z)+\gamma\}I_0(z)+\frac{\frac14z^2}{(1!)^2}+(1+\tfrac12)\frac{(\frac14z^2)^2}{(2!)^2}+(1+\tfrac12+\tfrac13)\frac{(\frac14z^2)^3}{(3!)^2}+\ldots$$

### Wronskians

**9.6.14**
$$W\{I_\nu(z),I_{-\nu}(z)\}=I_\nu(z)I_{-(\nu+1)}(z)-I_{\nu+1}(z)I_{-\nu}(z)=-2\sin(\nu\pi)/(\pi z)$$

**9.6.15**
$$W\{K_\nu(z),I_\nu(z)\}=I_\nu(z)K_{\nu+1}(z)+I_{\nu+1}(z)K_\nu(z)=1/z$$