9.2.15

$$R(\nu,z)\sim\sum_{k=0}^{\infty}(-)^k\frac{4\nu^2+16k^2-1}{4\nu^2-(4k-1)^2}\frac{(\nu,2k)}{(2z)^{2k}}$$
$$=1-\frac{(\mu-1)(\mu+15)}{2!(8z)^2}+\ldots$$

9.2.16

$$S(\nu,z)\sim\sum_{k=0}^{\infty}(-)^k\frac{4\nu^2+4(2k+1)^2-1}{4\nu^2-(4k+1)^2}\frac{(\nu,2k+1)}{(2z)^{2k+1}}$$
$$=\frac{\mu+3}{8z}-\frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8z)^3}+\ldots$$

Modulus and Phase

For real ν and positive x

9.2.17

$$M_\nu=|H_\nu^{(1)}(x)|=\sqrt{\{J_\nu^2(x)+Y_\nu^2(x)\}}$$
$$\theta_\nu=\arg H_\nu^{(1)}(x)=\arctan\{Y_\nu(x)/J_\nu(x)\}$$

9.2.18

$$N_\nu=|H_\nu^{(1)\prime}(x)|=\sqrt{\{J_\nu'^2(x)+Y_\nu'^2(x)\}}$$
$$\varphi_\nu=\arg H_\nu^{(1)\prime}(x)=\arctan\{Y_\nu'(x)/J_\nu'(x)\}$$

9.2.19 $J_\nu(x)=M_\nu\cos\theta_\nu,\qquad Y_\nu(x)=M_\nu\sin\theta_\nu,$

9.2.20 $J_\nu'(x)=N_\nu\cos\varphi_\nu,\qquad Y_\nu'(x)=N_\nu\sin\varphi_\nu.$

In the following relations, primes denote differentiations with respect to x.

9.2.21 $M_\nu^2\theta_\nu'=2/(\pi x)\qquad N_\nu^2\varphi_\nu'=2(x^2-\nu^2)/(\pi x^3)$

9.2.22 $N_\nu^2=M_\nu'^2+M_\nu^2\theta_\nu'^2=M_\nu'^2+4/(\pi x M_\nu)^2$

9.2.23 $(x^2-\nu^2)M_\nu M_\nu'+x^2N_\nu N_\nu'+xN_\nu^2=0$

9.2.24

$$\tan(\varphi_\nu-\theta_\nu)=M_\nu\theta_\nu'/M_\nu'=2/(\pi x M_\nu M_\nu')$$
$$M_\nu N_\nu\sin(\varphi_\nu-\theta_\nu)=2/(\pi x)$$

9.2.25 $x^2M_\nu''+xM_\nu'+(x^2-\nu^2)M_\nu-4/(\pi^2M_\nu^3)=0$

9.2.26

$$x^3w'''+x(4x^2+1-4\nu^2)w'+(4\nu^2-1)w=0,\quad w=xM_\nu^2$$

9.2.27 $$\theta_\nu'^2+\frac{1}{2}\frac{\theta_\nu'''}{\theta_\nu'}-\frac{3}{4}\left(\frac{\theta_\nu''}{\theta_\nu'}\right)^2=1-\frac{\nu^2-\frac{1}{4}}{x^2}$$

Asymptotic Expansions of Modulus and Phase

When ν is fixed, x is large and positive, and $\mu=4\nu^2$

9.2.28

$$M_\nu^2\sim\frac{2}{\pi x}\left\{1+\frac{1}{2}\frac{\mu-1}{(2x)^2}+\frac{1\cdot3}{2\cdot4}\frac{(\mu-1)(\mu-9)}{(2x)^4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{(\mu-1)(\mu-9)(\mu-25)}{(2x)^6}+\ldots\right\}$$

9.2.29

$$\theta_\nu\sim x-(\tfrac{1}{2}\nu+\tfrac{1}{4})\pi+\frac{\mu-1}{2(4x)}+\frac{(\mu-1)(\mu-25)}{6(4x)^3}+\frac{(\mu-1)(\mu^2-114\mu+1073)}{5(4x)^5}+\frac{(\mu-1)(5\mu^3-1535\mu^2+54703\mu-375733)}{14(4x)^7}+\ldots$$

9.2.30

$$N_\nu^2\sim\frac{2}{\pi x}\left\{1-\frac{1}{2}\frac{\mu-3}{(2x)^2}-\frac{1\cdot1}{2\cdot4}\frac{(\mu-1)(\mu-45)}{(2x)^4}-\ldots\right\}$$

The general term in the last expansion is given by

$$-\frac{1\cdot1\cdot3\ldots(2k-3)}{2\cdot4\cdot6\ldots(2k)}\times\frac{(\mu-1)(\mu-9)\ldots\{\mu-(2k-3)^2\}\{\mu-(2k+1)(2k-1)^2\}}{(2x)^{2k}}\ *$$

9.2.31

$$\phi_\nu\sim x-(\tfrac{1}{2}\nu-\tfrac{1}{4})\pi+\frac{\mu+3}{2(4x)}+\frac{\mu^2+46\mu-63}{6(4x)^3}+\frac{\mu^3+185\mu^2-2053\mu+1899}{5(4x)^5}+\ldots$$

If $\nu\geq0$, the remainder after k terms in **9.2.28** does not exceed the $(k+1)$th term in absolute value and is of the same sign, provided that $k>\nu-\frac{1}{2}$.

9.3. Asymptotic Expansions for Large Orders

Principal Asymptotic Forms

In the following equations it is supposed that $\nu\to\infty$ through real positive values, the other variables being fixed.

9.3.1

$$J_\nu(z)\sim\frac{1}{\sqrt{2\pi\nu}}\left(\frac{ez}{2\nu}\right)^\nu$$
$$Y_\nu(z)\sim-\sqrt{\frac{2}{\pi\nu}}\left(\frac{ez}{2\nu}\right)^{-\nu}$$

9.3.2

$$J_\nu(\nu\operatorname{sech}\alpha)\sim\frac{e^{\nu(\tanh\alpha-\alpha)}}{\sqrt{2\pi\nu\tanh\alpha}}\qquad(\alpha>0)$$
$$Y_\nu(\nu\operatorname{sech}\alpha)\sim-\frac{e^{\nu(\alpha-\tanh\alpha)}}{\sqrt{\frac{1}{2}\pi\nu\tanh\alpha}}\qquad(\alpha>0)$$

*See page II.