9.1.72

$$\lim \{\nu^{\mu} Q_{\nu}^{-\mu}\left(\cos \frac{x}{\nu}\right)\} = -\tfrac{1}{2}\pi Y_{\mu}(x) \qquad (x>0)$$

For $P_{\nu}^{-\mu}$ and $Q_{\nu}^{-\mu}$, see chapter **8.**

Continued Fractions

9.1.73

$$\frac{J_{\nu}(z)}{J_{\nu-1}(z)} = \frac{1}{2\nu z^{-1}-}\,\frac{1}{2(\nu+1)z^{-1}-}\,\frac{1}{2(\nu+2)z^{-1}-}\cdots$$
$$= \frac{\tfrac{1}{2}z/\nu}{1-}\,\frac{\tfrac{1}{4}z^2/\{\nu(\nu+1)\}}{1-}\,\frac{\tfrac{1}{4}z^2/\{(\nu+1)(\nu+2)\}}{1-}\cdots$$

Multiplication Theorem

9.1.74

$$\mathscr{C}_{\nu}(\lambda z) = \lambda^{\pm\nu} \sum_{k=0}^{\infty} \frac{(\mp)^k(\lambda^2-1)^k(\tfrac{1}{2}z)^k}{k!} \mathscr{C}_{\nu\pm k}(z) \qquad (|\lambda^2-1|<1)$$

If $\mathscr{C}=J$ and the upper signs are taken, the restriction on λ is unnecessary.

This theorem will furnish expansions of $\mathscr{C}_{\nu}(re^{i\theta})$ in terms of $\mathscr{C}_{\nu\pm k}(r)$.

Addition Theorems

Neumann's

9.1.75 $$\mathscr{C}_{\nu}(u\pm v) = \sum_{k=-\infty}^{\infty} \mathscr{C}_{\nu\mp k}(u) J_k(v) \qquad (|v|<|u|)$$

The restriction $|v|<|u|$ is unnecessary when $\mathscr{C}=J$ and ν is an integer or zero. Special cases are

9.1.76 $$1 = J_0^2(z) + 2\sum_{k=1}^{\infty} J_k^2(z)$$

9.1.77

$$0 = \sum_{k=0}^{2n} (-)^k J_k(z) J_{2n-k}(z) + 2\sum_{k=1}^{\infty} J_k(z) J_{2n+k}(z) \quad (n\geq 1)$$

9.1.78

$$J_n(2z) = \sum_{k=0}^{n} J_k(z) J_{n-k}(z) + 2\sum_{k=1}^{\infty} (-)^k J_k(z) J_{n+k}(z)$$

Graf's

9.1.79

$$\mathscr{C}_{\nu}(w) \,{\cos \atop \sin}\, \nu\chi = \sum_{k=-\infty}^{\infty} \mathscr{C}_{\nu+k}(u) J_k(v) \,{\cos \atop \sin}\, k\alpha \quad (|ve^{\pm i\alpha}|<|u|)$$

Gegenbauer's

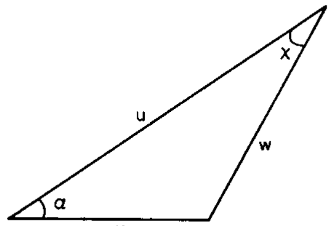
9.1.80

$$\frac{\mathscr{C}_{\nu}(w)}{w^{\nu}} = 2^{\nu}\Gamma(\nu) \sum_{k=0}^{\infty} (\nu+k) \frac{\mathscr{C}_{\nu+k}(u)}{u^{\nu}} \frac{J_{\nu+k}(v)}{v^{\nu}} C_k^{(\nu)}(\cos\alpha)$$
$$(\nu\neq 0, -1, \ldots, |ve^{\pm i\alpha}|<|u|)$$

In **9.1.79** and **9.1.80,**

$$w = \sqrt{(u^2+v^2-2uv\cos\alpha)},$$
$$u - v\cos\alpha = w\cos\chi, \quad v\sin\alpha = w\sin\chi$$

the branches being chosen so that $w\to u$ and $\chi\to 0$ as $v\to 0$. $C_k^{(\nu)}(\cos\alpha)$ is Gegenbauer's polynomial (see chapter **22**).



Gegenbauer's addition theorem.

If u, v are real and positive and $0\leq\alpha\leq\pi$, then w, χ are real and non-negative, and the geometrical relationship of the variables is shown in the diagram.

The restrictions $|ve^{\pm i\alpha}|<|u|$ are unnecessary in **9.1.79** when $\mathscr{C}=J$ and ν is an integer or zero, and in **9.1.80** when $\mathscr{C}=J$.

Degenerate Form ($u=\infty$):

9.1.81

$$e^{iv\cos\alpha} = \Gamma(\nu)(\tfrac{1}{2}v)^{-\nu} \sum_{k=0}^{\infty} (\nu+k) i^k J_{\nu+k}(v) C_k^{(\nu)}(\cos\alpha)$$
$$(\nu\neq 0, -1, \ldots)$$

Neumann's Expansion of an Arbitrary Function in a Series of Bessel Functions

9.1.82 $$f(z) = a_0 J_0(z) + 2\sum_{k=1}^{\infty} a_k J_k(z) \qquad (|z|<c)$$

where c is the distance of the nearest singularity of $f(z)$ from $z=0$,

9.1.83 $$a_k = \frac{1}{2\pi i}\int_{|z|=c'} f(t) O_k(t)\, dt \qquad (0<c'<c)$$

and $O_k(t)$ is Neumann's polynomial. The latter is defined by the generating function

9.1.84

$$\frac{1}{t-z} = J_0(z) O_0(t) + 2\sum_{k=1}^{\infty} J_k(z) O_k(t) \qquad (|z|<|t|)$$

$O_n(t)$ is a polynomial of degree $n+1$ in $1/t$; $O_0(t)=1/t$,

9.1.85

$$O_n(t) = \frac{1}{4}\sum_{k=0}^{\leq \frac{1}{2}n} \frac{n(n-k-1)!}{k!}\left(\frac{2}{t}\right)^{n-2k+1} \qquad (n=1,2,\ldots)$$

The more general form of expansion

9.1.86 $$f(z) = a_0 J_{\nu}(z) + 2\sum_{k=1}^{\infty} a_k J_{\nu+k}(z)$$