Other Differential Equations

9.1.49 $w''+\left(\lambda^2-\frac{\nu^2-\frac{1}{4}}{z^2}\right)w=0, \qquad w=z^{\frac{1}{2}}\mathscr{C}_\nu(\lambda z)$

9.1.50 $w''+\left(\frac{\lambda^2}{4z}-\frac{\nu^2-1}{4z^2}\right)w=0, \qquad w=z^{\frac{1}{2}}\mathscr{C}_\nu(\lambda z^{\frac{1}{2}})$

9.1.51 $w''+\lambda^2 z^{p-2}w=0, \qquad w=z^{\frac{1}{2}}\mathscr{C}_{1/p}(2\lambda z^{\frac{1}{2}p}/p)$

9.1.52

$$w''-\frac{2\nu-1}{z}w'+\lambda^2 w=0, \qquad w=z^\nu\mathscr{C}_\nu(\lambda z)$$

9.1.53

$$z^2w''+(1-2p)zw'+(\lambda^2q^2z^{2q}+p^2-\nu^2q^2)w=0, \qquad w=z^p\mathscr{C}_\nu(\lambda z^q)$$

9.1.54

$$w''+(\lambda^2e^{2z}-\nu^2)w=0, \qquad w=\mathscr{C}_\nu(\lambda e^z)$$

9.1.55

$$z^2(z^2-\nu^2)w''+z(z^2-3\nu^2)w' +\{(z^2-\nu^2)^2-(z^2+\nu^2)\}w=0, \qquad w=\mathscr{C}_\nu'(z)$$

9.1.56

$$w^{(2n)}=(-)^n\lambda^{2n}z^{-n}w, \qquad w=z^{\frac{1}{2}n}\mathscr{C}_n(2\lambda\alpha z^{\frac{1}{2}})$$

where α is any of the $2n$ roots of unity.

Differential Equations for Products

In the following $\vartheta\equiv z\frac{d}{dz}$ and $\mathscr{C}_\nu(z)$, $\mathscr{D}_\mu(z)$ are any cylinder functions of orders ν, μ respectively.

9.1.57

$$\{\vartheta^4-2(\nu^2+\mu^2)\vartheta^2+(\nu^2-\mu^2)^2\}w +4z^2(\vartheta+1)(\vartheta+2)w=0, \qquad w=\mathscr{C}_\nu(z)\mathscr{D}_\mu(z)$$

9.1.58

$$\vartheta(\vartheta^2-4\nu^2)w+4z^2(\vartheta+1)w=0, \qquad w=\mathscr{C}_\nu(z)\mathscr{D}_\nu(z)$$

9.1.59

$$z^3w'''+z(4z^2+1-4\nu^2)w'+(4\nu^2-1)w=0, \qquad w=z\mathscr{C}_\nu(z)\mathscr{D}_\nu(z)$$

Upper Bounds

9.1.60 $|J_\nu(x)|\le 1 \ (\nu\ge 0), \quad |J_\nu(x)|\le 1/\sqrt{2} \qquad (\nu\ge 1)$

9.1.61 $0<J_\nu(\nu)<\dfrac{2^{\frac{1}{3}}}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})\nu^{\frac{1}{3}}} \qquad (\nu>0)$

9.1.62 $|J_\nu(z)|\le\dfrac{|\frac{1}{2}z|^\nu e^{|\mathscr{I}z|}}{\Gamma(\nu+1)} \qquad (\nu\ge-\frac{1}{2})$ *

9.1.63 $|J_n(nz)|\le\left|\dfrac{z^n\exp\{n\sqrt{(1-z^2)}\}}{\{1+\sqrt{(1-z^2)}\}^n}\right|$

Derivatives With Respect to Order

9.1.64

$$\frac{\partial}{\partial\nu}J_\nu(z)=J_\nu(z)\ln(\tfrac{1}{2}z) -(\tfrac{1}{2}z)^\nu\sum_{k=0}^\infty(-)^k\frac{\psi(\nu+k+1)}{\Gamma(\nu+k+1)}\frac{(\frac{1}{4}z^2)^k}{k!}$$

9.1.65

$$\frac{\partial}{\partial\nu}Y_\nu(z)=\cot(\nu\pi)\left\{\frac{\partial}{\partial\nu}J_\nu(z)-\pi Y_\nu(z)\right\} -\csc(\nu\pi)\frac{\partial}{\partial\nu}J_{-\nu}(z)-\pi J_\nu(z) \qquad (\nu\ne 0,\pm1,\pm2,\ldots)$$

9.1.66

$$\left[\frac{\partial}{\partial\nu}J_\nu(z)\right]_{\nu=n}=\frac{\pi}{2}Y_n(z)+\frac{n!(\frac{1}{2}z)^{-n}}{2}\sum_{k=0}^{n-1}\frac{(\frac{1}{2}z)^kJ_k(z)}{(n-k)k!}$$

9.1.67

$$\left[\frac{\partial}{\partial\nu}Y_\nu(z)\right]_{\nu=n}=-\frac{\pi}{2}J_n(z)+\frac{n!(\frac{1}{2}z)^{-n}}{2}\sum_{k=0}^{n-1}\frac{(\frac{1}{2}z)^kY_k(z)}{(n-k)k!}$$

9.1.68

$$\left[\frac{\partial}{\partial\nu}J_\nu(z)\right]_{\nu=0}=\frac{\pi}{2}Y_0(z), \quad \left[\frac{\partial}{\partial\nu}Y_\nu(z)\right]_{\nu=0}=-\frac{\pi}{2}J_0(z)$$

Expressions in Terms of Hypergeometric Functions

9.1.69

$$J_\nu(z)=\frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\,{}_0F_1(\nu+1;-\tfrac{1}{4}z^2) =\frac{(\frac{1}{2}z)^\nu e^{-iz}}{\Gamma(\nu+1)}M(\nu+\tfrac{1}{2},2\nu+1,2iz)$$

9.1.70

$$J_\nu(z)=\frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\lim F\left(\lambda,\mu;\nu+1;-\frac{z^2}{4\lambda\mu}\right)$$

as λ, $\mu\to\infty$ through real or complex values; z, ν being fixed.

(${}_0F_1$ is the generalized hypergeometric function. For $M(a,b,z)$ and $F(a,b;c;z)$ see chapters **13** and **15**.)

Connection With Legendre Functions

If μ and x are fixed and $\nu\to\infty$ through real positive values

9.1.71

$$\lim\left\{\nu^\mu P_\nu^{-\mu}\left(\cos\frac{x}{\nu}\right)\right\}=J_\mu(x) \qquad (x>0)$$

*See page II.