Recurrence Relations

9.1.27

$$\mathscr{C}_{\nu-1}(z)+\mathscr{C}_{\nu+1}(z)=\frac{2\nu}{z}\mathscr{C}_\nu(z)$$

$$\mathscr{C}_{\nu-1}(z)-\mathscr{C}_{\nu+1}(z)=2\mathscr{C}'_\nu(z)$$

$$\mathscr{C}'_\nu(z)=\mathscr{C}_{\nu-1}(z)-\frac{\nu}{z}\mathscr{C}_\nu(z)$$

$$\mathscr{C}'_\nu(z)=-\mathscr{C}_{\nu+1}(z)+\frac{\nu}{z}\mathscr{C}_\nu(z)$$

$\mathscr{C}$ denotes $J, Y, H^{(1)}, H^{(2)}$ or any linear combination of these functions, the coefficients in which are independent of z and ν.

9.1.28 $J'_0(z)=-J_1(z) \qquad Y'_0(z)=-Y_1(z)$

If $f_\nu(z)=z^p\mathscr{C}_\nu(\lambda z^q)$ where p, q, λ are independent of ν, then

9.1.29

$$f_{\nu-1}(z)+f_{\nu+1}(z)=(2\nu/\lambda)z^{-q}f_\nu(z)$$

$$(p+\nu q)f_{\nu-1}(z)+(p-\nu q)f_{\nu+1}(z)=(2\nu/\lambda)z^{1-q}f'_\nu(z)$$

$$zf'_\nu(z)=\lambda qz^qf_{\nu-1}(z)+(p-\nu q)f_\nu(z)$$

$$zf'_\nu(z)=-\lambda qz^qf_{\nu+1}(z)+(p+\nu q)f_\nu(z)$$

Formulas for Derivatives

9.1.30

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k\{z^\nu\mathscr{C}_\nu(z)\}=z^{\nu-k}\mathscr{C}_{\nu-k}(z)$$

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k\{z^{-\nu}\mathscr{C}_\nu(z)\}=(-)^kz^{-\nu-k}\mathscr{C}_{\nu+k}(z)$$

$$(k=0,1,2,\ldots)$$

9.1.31

$$\mathscr{C}_\nu^{(k)}(z)=\frac{1}{2^k}\left\{\mathscr{C}_{\nu-k}(z)-\binom{k}{1}\mathscr{C}_{\nu-k+2}(z)+\binom{k}{2}\mathscr{C}_{\nu-k+4}(z)-\ldots+(-)^k\mathscr{C}_{\nu+k}(z)\right\}$$

$$(k=0,1,2,\ldots)$$

Recurrence Relations for Cross-Products

If

9.1.32

$$p_\nu=J_\nu(a)Y_\nu(b)-J_\nu(b)Y_\nu(a)$$
$$q_\nu=J_\nu(a)Y'_\nu(b)-J'_\nu(b)Y_\nu(a)$$
$$r_\nu=J'_\nu(a)Y_\nu(b)-J_\nu(b)Y'_\nu(a)$$
$$s_\nu=J'_\nu(a)Y'_\nu(b)-J'_\nu(b)Y'_\nu(a)$$

then

9.1.33

$$p_{\nu+1}-p_{\nu-1}=-\frac{2\nu}{a}q_\nu-\frac{2\nu}{b}r_\nu$$

$$q_{\nu+1}+r_\nu=\frac{\nu}{a}p_\nu-\frac{\nu+1}{b}p_{\nu+1}$$

$$r_{\nu+1}+q_\nu=\frac{\nu}{b}p_\nu-\frac{\nu+1}{a}p_{\nu+1}$$

$$s_\nu=\frac{1}{2}p_{\nu+1}+\frac{1}{2}p_{\nu-1}-\frac{\nu^2}{ab}p_\nu$$

and

9.1.34 $$p_\nu s_\nu-q_\nu r_\nu=\frac{4}{\pi^2ab}$$

Analytic Continuation

In **9.1.35** to **9.1.38**, m is an integer.

9.1.35 $$J_\nu(ze^{m\pi i})=e^{m\nu\pi i}J_\nu(z)$$

9.1.36

$$Y_\nu(ze^{m\pi i})=e^{-m\nu\pi i}Y_\nu(z)+2i\sin(m\nu\pi)\cot(\nu\pi)J_\nu(z)$$

9.1.37

$$\sin(\nu\pi)H_\nu^{(1)}(ze^{m\pi i})=-\sin\{(m-1)\nu\pi\}H_\nu^{(1)}(z)-e^{-\nu\pi i}\sin(m\nu\pi)H_\nu^{(2)}(z)$$

9.1.38

$$\sin(\nu\pi)H_\nu^{(2)}(ze^{m\pi i})=\sin\{(m+1)\nu\pi\}H_\nu^{(2)}(z)+e^{\nu\pi i}\sin(m\nu\pi)H_\nu^{(1)}(z)$$

9.1.39

$$H_\nu^{(1)}(ze^{\pi i})=-e^{-\nu\pi i}H_\nu^{(2)}(z)$$
$$H_\nu^{(2)}(ze^{-\pi i})=-e^{\nu\pi i}H_\nu^{(1)}(z)$$

9.1.40

$$J_\nu(\bar{z})=\overline{J_\nu(z)} \qquad Y_\nu(\bar{z})=\overline{Y_\nu(z)}$$

$$H_\nu^{(1)}(\bar{z})=\overline{H_\nu^{(2)}(z)} \qquad H_\nu^{(2)}(\bar{z})=\overline{H_\nu^{(1)}(z)} \qquad (\nu \text{ real})$$

Generating Function and Associated Series

9.1.41 $$e^{\frac{1}{2}z(t-1/t)}=\sum_{k=-\infty}^{\infty}t^kJ_k(z) \qquad (t\neq 0)$$

9.1.42 $$\cos(z\sin\theta)=J_0(z)+2\sum_{k=1}^{\infty}J_{2k}(z)\cos(2k\theta)$$

9.1.43 $$\sin(z\sin\theta)=2\sum_{k=0}^{\infty}J_{2k+1}(z)\sin\{(2k+1)\theta\}$$

9.1.44

$$\cos(z\cos\theta)=J_0(z)+2\sum_{k=1}^{\infty}(-)^kJ_{2k}(z)\cos(2k\theta)$$

9.1.45

$$\sin(z\cos\theta)=2\sum_{k=0}^{\infty}(-)^kJ_{2k+1}(z)\cos\{(2k+1)\theta\}$$

9.1.46 $$1=J_0(z)+2J_2(z)+2J_4(z)+2J_6(z)+\ldots$$

9.1.47

$$\cos z=J_0(z)-2J_2(z)+2J_4(z)-2J_6(z)+\ldots$$

9.1.48 $$\sin z=2J_1(z)-2J_3(z)+2J_5(z)-\ldots$$