

Limiting Forms for Small Arguments

When ν is fixed and $z \rightarrow 0$

9.1.7

$$J_\nu(z) \sim (\frac{1}{2}z)^\nu / \Gamma(\nu+1) \quad (\nu \neq -1, -2, -3, \dots)$$

9.1.8 $Y_0(z) \sim -iH_0^{(1)}(z) \sim iH_0^{(2)}(z) \sim (2/\pi) \ln z$

9.1.9

$$Y_\nu(z) \sim -iH_\nu^{(1)}(z) \sim iH_\nu^{(2)}(z) \sim -(1/\pi)\Gamma(\nu)(\frac{1}{2}z)^{-\nu} \quad (\Re \nu > 0)$$

Ascending Series

9.1.10 $J_\nu(z) = (\frac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}z^2)^k}{k! \Gamma(\nu+k+1)}$

9.1.11

$$Y_n(z) = -\frac{(\frac{1}{2}z)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} (\frac{1}{4}z^2)^k + \frac{2}{\pi} \ln(\frac{1}{2}z) J_n(z) - \frac{(\frac{1}{2}z)^n}{\pi} \sum_{k=0}^{\infty} \{ \psi(k+1) + \psi(n+k+1) \} \frac{(-\frac{1}{4}z^2)^k}{k!(n+k)!}$$

where $\psi(n)$ is given by 6.3.2.

9.1.12 $J_0(z) = 1 - \frac{\frac{1}{4}z^2}{(1!)^2} + \frac{(\frac{1}{4}z^2)^2}{(2!)^2} - \frac{(\frac{1}{4}z^2)^3}{(3!)^2} + \dots$

9.1.13

$$Y_0(z) = \frac{2}{\pi} \{ \ln(\frac{1}{2}z) + \gamma \} J_0(z) + \frac{2}{\pi} \{ \frac{\frac{1}{4}z^2}{(1!)^2} - (1+\frac{1}{2}) \frac{(\frac{1}{4}z^2)^2}{(2!)^2} + (1+\frac{1}{2}+\frac{1}{3}) \frac{(\frac{1}{4}z^2)^3}{(3!)^2} - \dots \}$$

9.1.14

$$J_\nu(z) J_\mu(z) = (\frac{1}{2}z)^{\nu+\mu} \sum_{k=0}^{\infty} \frac{(-)^k \Gamma(\nu+\mu+2k+1) (\frac{1}{4}z^2)^k}{\Gamma(\nu+k+1) \Gamma(\mu+k+1) \Gamma(\nu+\mu+k+1) k!}$$

Wronskians

9.1.15

$$W\{J_\nu(z), J_{-\nu}(z)\} = J_{\nu+1}(z) J_{-\nu}(z) + J_\nu(z) J_{-(\nu+1)}(z) = -2 \sin(\nu\pi) / (\pi z)$$

9.1.16

$$W\{J_\nu(z), Y_\nu(z)\} = J_{\nu+1}(z) Y_\nu(z) - J_\nu(z) Y_{\nu+1}(z) = 2 / (\pi z)$$

9.1.17

$$W\{H_\nu^{(1)}(z), H_\nu^{(2)}(z)\} = H_{\nu+1}^{(1)}(z) H_\nu^{(2)}(z) - H_\nu^{(1)}(z) H_{\nu+1}^{(2)}(z) = -4i / (\pi z)$$

Integral Representations

9.1.18

$$J_0(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta) d\theta = \frac{1}{\pi} \int_0^\pi \cos(z \cos \theta) d\theta$$

9.1.19

$$Y_0(z) = \frac{4}{\pi^2} \int_0^{\frac{1}{2}\pi} \cos(z \cos \theta) \{ \gamma + \ln(2z \sin^2 \theta) \} d\theta$$

9.1.20

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}} \Gamma(\nu + \frac{1}{2})} \int_0^\pi \cos(z \cos \theta) \sin^{2\nu} \theta d\theta = \frac{2(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}} \Gamma(\nu + \frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos(zt) dt \quad (\Re \nu > -\frac{1}{2})$$

9.1.21

$$J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta - n\theta) d\theta = \frac{i^{-n}}{\pi} \int_0^\pi e^{iz \cos \theta} \cos(n\theta) d\theta$$

9.1.22

$$J_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(z \sin \theta - \nu\theta) d\theta - \frac{\sin(\nu\pi)}{\pi} \int_0^\infty e^{-z \sinh t - \nu t} dt \quad (|\arg z| < \frac{1}{2}\pi)$$

$$Y_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(z \sin \theta - \nu\theta) d\theta - \frac{1}{\pi} \int_0^\infty \{ e^{\nu t} + e^{-\nu t} \cos(\nu\pi) \} e^{-z \sinh t} dt \quad (|\arg z| < \frac{1}{2}\pi)$$

9.1.23

$$J_0(x) = \frac{2}{\pi} \int_0^\infty \sin(x \cosh t) dt \quad (x > 0)$$

$$Y_0(x) = -\frac{2}{\pi} \int_0^\infty \cos(x \cosh t) dt \quad (x > 0)$$

9.1.24

$$J_\nu(x) = \frac{2(\frac{1}{2}x)^{-\nu}}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2}-\nu)} \int_1^\infty \frac{\sin(xt) dt}{(t^2-1)^{\nu+\frac{1}{2}}} \quad (|\Re \nu| < \frac{1}{2}, x > 0)$$

$$Y_\nu(x) = -\frac{2(\frac{1}{2}x)^{-\nu}}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2}-\nu)} \int_1^\infty \frac{\cos(xt) dt}{(t^2-1)^{\nu+\frac{1}{2}}} \quad (|\Re \nu| < \frac{1}{2}, x > 0)$$

9.1.25

$$H_\nu^{(1)}(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty+\pi i} e^{z \sinh t - \nu t} dt \quad (|\arg z| < \frac{1}{2}\pi)$$

$$H_\nu^{(2)}(z) = -\frac{1}{\pi i} \int_{-\infty}^{\infty-\pi i} e^{z \sinh t - \nu t} dt \quad (|\arg z| < \frac{1}{2}\pi)$$

9.1.26

$$J_\nu(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(-t) (\frac{1}{2}x)^{\nu+2t}}{\Gamma(\nu+t+1)} dt \quad (\Re \nu > 0, x > 0)$$

In the last integral the path of integration must lie to the left of the points $t=0, 1, 2, \dots$