## 8.12. Conical Functions

$(P^{\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta),\ Q^{\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta))$

(Only special properties are given as other properties and representations follow from earlier sections with $\nu=-\frac{1}{2}+i\lambda$ ($\lambda$, a real parameter) and $z=\cos\theta$.)

**8.12.1**

$$P_{-\frac{1}{2}+i\lambda}(\cos\theta)=1+\frac{4\lambda^2+1^2}{2^2}\sin^2\frac{\theta}{2}+\frac{(4\lambda^2+1^2)(4\lambda^2+3^2)}{2^24^2}\sin^4\frac{\theta}{2}+\ldots \qquad (0\leq\theta<\pi)$$

**8.12.2** $$P_{-\frac{1}{2}+i\lambda}(\cos\theta)=P_{-\frac{1}{2}-i\lambda}(\cos\theta)$$

**8.12.3** $$P_{-\frac{1}{2}+i\lambda}(\cos\theta)=\frac{2}{\pi}\int_0^\theta\frac{\cosh\lambda t\,dt}{\sqrt{2(\cos t-\cos\theta)}}$$

**8.12.4**

$$Q_{-\frac{1}{2}\mp i\lambda}(\cos\theta)=\pm i\sinh\lambda\pi\int_0^\infty\frac{\cos\lambda t\,dt}{\sqrt{2(\cosh t+\cos\theta)}}+\int_0^\infty\frac{\cosh\lambda t\,dt}{\sqrt{2(\cosh t-\cos\theta)}}$$

**8.12.5**

$$P_{-\frac{1}{2}+i\lambda}(-\cos\theta)=\frac{\cosh\lambda\pi}{\pi}[Q_{-\frac{1}{2}+i\lambda}(\cos\theta)+Q_{-\frac{1}{2}-i\lambda}(\cos\theta)]$$

## * 8.13. Relation to Elliptic Integrals (see chapter 17) ($\mathscr{R}\eta>0$)

**8.13.1** $$P_{-\frac{1}{2}}(z)=\frac{2}{\pi}\sqrt{\frac{2}{z+1}}\,K\left(\sqrt{\frac{z-1}{z+1}}\right)$$

**8.13.2** $$P_{-\frac{1}{2}}(\cosh\eta)=\left[\frac{\pi}{2}\cosh\frac{\eta}{2}\right]^{-1}K\left(\tanh\frac{\eta}{2}\right)$$

**8.13.3** $$Q_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{z+1}}\,K\left(\sqrt{\frac{2}{z+1}}\right)$$

**8.13.4** $$Q_{-\frac{1}{2}}(\cosh\eta)=2e^{-\eta/2}K(e^{-\eta})$$

**8.13.5**

$$P_{\frac{1}{2}}(z)=\frac{2}{\pi}(z+\sqrt{z^2-1})^{\frac{1}{2}}E\left(\sqrt{\frac{2(z^2-1)^{1/2}}{z+(z^2-1)^{1/2}}}\right)$$

**8.13.6** $$P_{\frac{1}{2}}(\cosh\eta)=\frac{2}{\pi}e^{\eta/2}E(\sqrt{1-e^{-2\eta}})$$

**8.13.7**

$$Q_{\frac{1}{2}}(z)=z\sqrt{\frac{2}{z+1}}\,K\left(\sqrt{\frac{2}{z+1}}\right)-[2(z+1)]^{\frac{1}{2}}E\left(\sqrt{\frac{2}{z+1}}\right)$$ *

$(-1<x<1)$

**8.13.8** $$P_{-\frac{1}{2}}(x)=\frac{2}{\pi}K\left(\sqrt{\frac{1-x}{2}}\right)$$

**8.13.9** $$P_{-\frac{1}{2}}(\cos\theta)=\frac{2}{\pi}K\left(\sin\frac{\theta}{2}\right)$$

**8.13.10** $$Q_{-\frac{1}{2}}(x)=K\left(\sqrt{\frac{1+x}{2}}\right)$$ *

**8.13.11** $$P_{\frac{1}{2}}(x)=\frac{2}{\pi}\left[2E\left(\sqrt{\frac{1-x}{2}}\right)-K\left(\sqrt{\frac{1-x}{2}}\right)\right]$$

**8.13.12** $$Q_{\frac{1}{2}}(x)=K\left(\sqrt{\frac{1+x}{2}}\right)-2E\left(\sqrt{\frac{1+x}{2}}\right)$$ *

## 8.14. Integrals

**8.14.1** $$\int_1^\infty P_\nu(x)Q_\rho(x)dx=[(\rho-\nu)(\rho+\nu+1)]^{-1} \qquad (\mathscr{R}\rho>\mathscr{R}\nu>0)$$

**8.14.2** $$\int_1^\infty Q_\nu(x)Q_\rho(x)dx=[(\rho-\nu)(\rho+\nu+1)]^{-1}[\psi(\rho+1)-\psi(\nu+1)]$$
$(\mathscr{R}(\rho+\nu)>-1,\ \rho+\nu+1\neq0;\ \nu,\ \rho\neq-1,\ -2,\ -3,\ \ldots)$

**8.14.3** $$\int_1^\infty[Q_\nu(x)]^2dx=(2\nu+1)^{-1}\psi'(\nu+1) \qquad (\mathscr{R}\nu>-\tfrac{1}{2})$$

**8.14.4** $$\int_{-1}^1 P_\nu(x)P_\rho(x)dx=\frac{2}{\pi^2}[(\rho-\nu)(\rho+\nu+1)]^{-1}\{2\sin\pi\nu\sin\pi\rho[\psi(\nu+1)-\psi(\rho+1)]+\pi\sin(\pi\rho-\pi\nu)\}$$
$(\rho+\nu+1\neq0)$

**8.14.5** $$\int_{-1}^1[P_\nu(x)]^2dx=\frac{\pi^2-2(\sin\pi\nu)^2\psi'(\nu+1)}{\pi^2(\nu+\frac{1}{2})}$$ *

**8.14.6** $$\int_{-1}^1 Q_\nu(x)Q_\rho(x)dx=[(\rho-\nu)(\rho+\nu+1)]^{-1}\{[\psi(\nu+1)-\psi(\rho+1)][1+\cos\rho\pi\cos\nu\pi]-\tfrac{1}{2}\pi\sin(\nu\pi-\rho\pi)\}$$ *
$(\rho+\nu+1\neq0;\ \nu,\ \rho\neq-1,\ -2,\ -3,\ \ldots)$

**8.14.7** $$\int_{-1}^1[Q_\nu(x)]^2dx=(2\nu+1)^{-1}\{\tfrac{1}{2}\pi^2-\psi'(\nu+1)[1+(\cos\nu\pi)^2]\} \qquad (\nu\neq-1,\ -2,\ -3,\ldots)$$

*See page II.