6.5.30

$$\gamma(a, x+y)-\gamma(a, x)$$
$$=e^{-x}x^{a-1}\sum_{n=0}^{\infty}\frac{(a-1)(a-2)\ldots(a-n)}{x^n}[1-e^{-y}e_n(y)]$$
$$(|y|<|x|)$$

Continued Fraction

6.5.31

$$\Gamma(a,x)=e^{-x}x^a\left(\frac{1}{x+}\,\frac{1-a}{1+}\,\frac{1}{x+}\,\frac{2-a}{1+}\,\frac{2}{x+}\cdots\right)$$
$$(x>0, |a|<\infty)$$

Asymptotic Expansions

6.5.32

$$\Gamma(a, z)\sim z^{a-1}e^{-z}\left[1+\frac{a-1}{z}+\frac{(a-1)(a-2)}{z^2}+\cdots\right]$$
$$\left(z\to\infty \text{ in } |\arg z|<\frac{3\pi}{2}\right)$$

Suppose $R_n(a,z)=u_{n+1}(a,z)+\ldots$ is the remainder after n terms in this series. Then if a,z are real, we have for $n>a-2$

$$|R_n(a,z)|\leq|u_{n+1}(a,z)|$$

and sign $R_n(a,z)=$ sign $u_{n+1}(a,z)$.

6.5.33 $$\gamma(a,z)\sim\sum_{n=0}^{\infty}\frac{(-1)^n z^{a+n}}{(a+n)n!}\qquad(a\to+\infty)$$

6.5.34 $$\lim_{n\to\infty}\frac{e_n(\alpha n)}{e^{\alpha n}}=\begin{cases}0 \text{ for } \alpha>1\\ \frac{1}{2} \text{ for } \alpha=1\\ 1 \text{ for } 0\leq\alpha<1\end{cases}$$

6.5.35

$$\Gamma(z+1,z)\sim e^{-z}z^z\left(\sqrt{\frac{\pi}{2}}z^{\frac{1}{2}}+\frac{2}{3}+\frac{\sqrt{2\pi}}{24}\frac{1}{z^{\frac{1}{2}}}+\cdots\right)$$
$$(z\to\infty \text{ in } |\arg z|<\tfrac{1}{2}\pi)$$

Definite Integrals

6.5.36

$$\int_0^{\infty}e^{-at}\,\Gamma(b,ct)\,dt=\frac{\Gamma(b)}{a}\left[1-\frac{c^b}{(a+c)^b}\right] \quad *$$
$$(\mathscr{R}(a+c)>0, \mathscr{R}b>-1)$$

6.5.37

$$\int_0^{\infty}t^{a-1}\Gamma(b,t)\,dt=\frac{\Gamma(a+b)}{a}$$
$$(\mathscr{R}(a+b)>0,\quad \mathscr{R}a>0)$$

6.6. Incomplete Beta Function

6.6.1 $$B_x(a,b)=\int_0^x t^{a-1}(1-t)^{b-1}dt$$

6.6.2 $$I_x(a,b)=B_x(a,b)/B(a,b)$$

For statistical applications, see **26.5**.

Symmetry

6.6.3 $$I_x(a,b)=1-I_{1-x}(b,a)$$

Relation to Binomial Expansion

6.6.4 $$I_p(a,n-a+1)=\sum_{j=a}^{n}\binom{n}{j}p^j(1-p)^{n-j}$$

For binomial distribution, see **26.1**.

Recurrence Formulas

6.6.5 $$I_x(a,b)=xI_x(a-1,b)+(1-x)I_x(a,b-1)$$

6.6.6 $$(a+b-ax)I_x(a,b)=a(1-x)I_x(a+1,b-1)+bI_x(a,b+1) \quad *$$

6.6.7 $$(a+b)I_x(a,b)=aI_x(a+1,b)+bI_x(a,b+1)$$

Relation to Hypergeometric Function

6.6.8 $$B_x(a,b)=a^{-1}x^aF(a,1-b;a+1;x)$$

Numerical Methods

6.7. Use and Extension of the Tables

Example 1. Compute $\Gamma(6.38)$ to 8S. Using the recurrence relation **6.1.16** and **Table 6.1** we have,

$$\Gamma(6.38)=[(5.38)(4.38)(3.38)(2.38)(1.38)]\Gamma(1.38)$$
$$=232.43671.$$

Example 2. Compute $\ln\Gamma(56.38)$, using **Table 6.4** and linear interpolation in f_2. We have

$$\ln\Gamma(56.38)=(56.38-\tfrac{1}{2})\ln(56.38)-(56.38)+f_2(56.38)$$

The error of linear interpolation in the table of the function f_2 is smaller than 10^{-7} in this region. Hence, $f_2(56.38)=.92041\ 67$ and $\ln\Gamma(56.38)=169.85497\ 42$.

Direct interpolation in **Table 6.4** of $\log_{10}\Gamma(n)$ eliminates the necessity of employing logarithms. However, the error of linear interpolation is .002 so that $\log_{10}\Gamma(n)$ is obtained with a relative error of 10^{-5}.

*See page II.