

6.4. Polygamma Functions⁷

6.4.1

$$\psi^{(n)}(z) = \frac{d^n}{dz^n} \psi(z) = \frac{d^{n+1}}{dz^{n+1}} \ln \Gamma(z) \quad (n=1,2,3, \dots)$$

$$* \quad = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-zt}}{1-e^{-t}} dt \quad (\Re z > 0)$$

$\psi^{(n)}(z), (n=0,1, \dots)$, is a single valued analytic function over the entire complex plane save at the points $z = -m (m=0,1,2, \dots)$ where it possesses poles of order $(n+1)$.

Integer Values

6.4.2

$$\psi^{(n)}(1) = (-1)^{n+1} n! \zeta(n+1) \quad (n=1,2,3, \dots)$$

6.4.3

$$\psi^{(m)}(n+1) = (-1)^m m! \left[-\zeta(m+1) + 1 + \frac{1}{2^{m+1}} + \dots + \frac{1}{n^{m+1}} \right]$$

Fractional Values

6.4.4

$$\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^{n+1} n! (2^{n+1} - 1) \zeta(n+1) \quad (n=1,2, \dots)$$

6.4.5 $\psi'(n + \frac{1}{2}) = \frac{1}{2} \pi^2 - 4 \sum_{k=1}^n (2k-1)^{-2}$

Recurrence Formula

6.4.6 $\psi^{(n)}(z+1) = \psi^{(n)}(z) + (-1)^n n! z^{-n-1}$

Reflection Formula

6.4.7

$$\psi^{(n)}(1-z) + (-1)^{n+1} \psi^{(n)}(z) = (-1)^n \pi \frac{d^n}{dz^n} \cot \pi z$$

Multiplication Formula

6.4.8

$$* \quad \psi^{(n)}(mz) = \delta \ln m + \frac{1}{m^{n+1}} \sum_{k=0}^{m-1} \psi^{(n)}\left(z + \frac{k}{m}\right)$$

$$\delta = 1, \quad n = 0$$

$$\delta = 0, \quad n > 0$$

⁷ ψ' is known as the trigamma function. $\psi'', \psi^{(3)}, \psi^{(4)}$ are the tetra-, penta-, and hexagamma functions respectively. Some authors write $\psi(z) = d[\ln \Gamma(z+1)]/dz$, and similarly for the polygamma functions.

* See page 11.

Series Expansions

6.4.9

$$\psi^{(n)}(1+z) = (-1)^{n+1} \left[n! \zeta(n+1) - \frac{(n+1)!}{1!} \zeta(n+2)z + \frac{(n+2)!}{2!} \zeta(n+3)z^2 - \dots \right] \quad (|z| < 1)$$

6.4.10

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^\infty (z+k)^{-n-1} \quad (z \neq 0, -1, -2, \dots)$$

Asymptotic Formulas

6.4.11

$$\psi^{(n)}(z) \sim (-1)^{n-1} \left[\frac{(n-1)!}{z^n} + \frac{n!}{2z^{n+1}} + \sum_{k=1}^\infty B_{2k} \frac{(2k+n-1)!}{(2k)! z^{2k+n}} \right] \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.4.12

$$\psi'(z) \sim \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} - \frac{1}{30z^5} + \frac{1}{42z^7} - \frac{1}{30z^9} + \dots \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.4.13

$$\psi''(z) \sim -\frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{2z^4} + \frac{1}{6z^6} - \frac{1}{6z^8} + \frac{3}{10z^{10}} - \frac{5}{6z^{12}} + \dots \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.4.14

$$\psi^{(3)}(z) \sim \frac{2}{z^3} + \frac{3}{z^4} + \frac{2}{z^5} - \frac{1}{z^7} + \frac{4}{3z^9} - \frac{3}{z^{11}} + \frac{10}{z^{13}} - \dots \quad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.5. Incomplete Gamma Function (see also 26.4)

6.5.1

$$P(a, x) = \frac{1}{\Gamma(a)} \int_0^x e^{-t} t^{a-1} dt \quad (\Re a > 0)$$

6.5.2

$$\gamma(a, x) = P(a, x) \Gamma(a) = \int_0^x e^{-t} t^{a-1} dt \quad (\Re a > 0)$$

6.5.3

$$\Gamma(a, x) = \Gamma(a) - \gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt$$

6.5.4

$$\gamma^*(a, x) = x^{-a} P(a, x) = \frac{x^{-a}}{\Gamma(a)} \gamma(a, x)$$

γ^* is a single valued analytic function of a and x possessing no finite singularities.