Reflection Formula

6.3.7 $$\psi(1-z)=\psi(z)+\pi\cot\pi z$$

Duplication Formula

6.3.8 $$\psi(2z)=\tfrac{1}{2}\psi(z)+\tfrac{1}{2}\psi(z+\tfrac{1}{2})+\ln 2$$

Psi Function in the Complex Plane

6.3.9 $$\psi(\bar{z})=\overline{\psi(z)}$$

6.3.10
$$\mathscr{R}\psi(iy)=\mathscr{R}\psi(-iy)=\mathscr{R}\psi(1+iy)=\mathscr{R}\psi(1-iy)$$

6.3.11 $$\mathscr{I}\psi(iy)=\tfrac{1}{2}y^{-1}+\tfrac{1}{2}\pi\coth\pi y$$

6.3.12 $$\mathscr{I}\psi(\tfrac{1}{2}+iy)=\tfrac{1}{2}\pi\tanh\pi y$$

6.3.13 $$\mathscr{I}\psi(1+iy)=-\frac{1}{2y}+\tfrac{1}{2}\pi\coth\pi y$$
$$=y\sum_{n=1}^{\infty}(n^2+y^2)^{-1}$$

Series Expansions

6.3.14 $$\psi(1+z)=-\gamma+\sum_{n=2}^{\infty}(-1)^n\zeta(n)z^{n-1}\qquad(|z|<1)$$

6.3.15
$$\psi(1+z)=\tfrac{1}{2}z^{-1}-\tfrac{1}{2}\pi\cot\pi z-(1-z^2)^{-1}+1-\gamma$$
$$-\sum_{n=1}^{\infty}[\zeta(2n+1)-1]z^{2n}\qquad(|z|<2)$$

6.3.16
$$\psi(1+z)=-\gamma+\sum_{n=1}^{\infty}\frac{z}{n(n+z)}\qquad(z\neq-1,-2,-3,\ldots)$$

6.3.17
$$\mathscr{R}\psi(1+iy)=1-\gamma-\frac{1}{1+y^2}$$
$$+\sum_{n=1}^{\infty}(-1)^{n+1}[\zeta(2n+1)-1]y^{2n}$$
$$(|y|<2)$$
$$=-\gamma+y^2\sum_{n=1}^{\infty}n^{-1}(n^2+y^2)^{-1}$$
$$(-\infty<y<\infty)$$

Asymptotic Formulas

6.3.18
$$\psi(z)\sim\ln z-\frac{1}{2z}-\sum_{n=1}^{\infty}\frac{B_{2n}}{2nz^{2n}}$$
$$=\ln z-\frac{1}{2z}-\frac{1}{12z^2}+\frac{1}{120z^4}-\frac{1}{252z^6}+\ldots$$
$$(z\to\infty\text{ in }|\arg z|<\pi)$$

6.3.19
$$\mathscr{R}\psi(1+iy)\sim\ln y+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_{2n}}{2ny^{2n}}$$
$$=\ln y+\frac{1}{12y^2}+\frac{1}{120y^4}+\frac{1}{252y^6}+\ldots$$
$$(y\to\infty)$$

Extrema[6] of $\Gamma(x)$ — Zeros of $\psi(x)$

$$\Gamma'(x_n)=\psi(x_n)=0$$

n	x_n	$\Gamma(x_n)$
0	+1.462	+0.886
1	−0.504	−3.545
2	−1.573	+2.302
3	−2.611	−0.888
4	−3.635	+0.245
5	−4.653	−0.053
6	−5.667	+0.009
7	−6.678	−0.001

$x_0=1.46163\ 21449\ 68362$

$\Gamma(x_0)=.88560\ 31944\ 10889$

6.3.20 $$x_n=-n+(\ln n)^{-1}+o[(\ln n)^{-2}]$$

Definite Integrals

6.3.21
$$\psi(z)=\int_0^\infty\left[\frac{e^{-t}}{t}-\frac{e^{-zt}}{1-e^{-t}}\right]dt\qquad(\mathscr{R}z>0)$$
$$=\int_0^\infty\left[e^{-t}-\frac{1}{(1+t)^z}\right]\frac{dt}{t}$$
$$=\ln z-\frac{1}{2z}-2\int_0^\infty\frac{t\,dt}{(t^2+z^2)(e^{2\pi t}-1)}$$
$$\left(|\arg z|<\frac{\pi}{2}\right)$$

6.3.22
$$\psi(z)+\gamma=\int_0^\infty\frac{e^{-t}-e^{-zt}}{1-e^{-t}}dt=\int_0^1\frac{1-t^{z-1}}{1-t}dt$$
$$\gamma=\int_0^\infty\left(\frac{1}{e^t-1}-\frac{1}{te^t}\right)dt$$
$$=\int_0^\infty\left(\frac{1}{1+t}-e^{-t}\right)\frac{dt}{t}$$

[6] From W. Sibagaki, Theory and applications of the gamma function, Iwanami Syoten, Tokyo, Japan, 1952 (with permission).