Polynomial Approximations[3]

6.1.35 $\quad 0 \le x \le 1$

$$\Gamma(x+1)=x!=1+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\epsilon(x)$$

$$|\epsilon(x)|\le 5\times 10^{-5}$$

$a_1=-.57486\ 46$	$a_4=\ \ .42455\ 49$
$a_2=\ \ .95123\ 63$	$a_5=-.10106\ 78$
$a_3=-.69985\ 88$	

6.1.36 $\quad 0\le x\le 1$

$$\Gamma(x+1)=x!=1+b_1x+b_2x^2+\ .\ .\ .\ +b_8x^8+\epsilon(x)$$

$$|\epsilon(x)|\le 3\times 10^{-7}$$

$b_1=-.57719\ 1652$	$b_5=-.75670\ 4078$
$b_2=\ \ .98820\ 5891$	$b_6=\ \ .48219\ 9394$
$b_3=-.89705\ 6937$	$b_7=-.19352\ 7818$
$b_4=\ \ .91820\ 6857$	$b_8=\ \ .03586\ 8343$

Stirling's Formula

6.1.37

$$\Gamma(z)\sim e^{-z}z^{z-\frac{1}{2}}(2\pi)^{\frac{1}{2}}\left[1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51840z^3}-\frac{571}{2488320z^4}+\cdots\right]\qquad (z\to\infty \text{ in } |\arg z|<\pi)$$

6.1.38

$$x!=\sqrt{2\pi}\,x^{x+\frac{1}{2}}\exp\left(-x+\frac{\theta}{12x}\right)\qquad (x>0,\ 0<\theta<1)$$

Asymptotic Formulas

6.1.39

$$\Gamma(az+b)\sim\sqrt{2\pi}\,e^{-az}(az)^{az+b-\frac{1}{2}}\qquad (|\arg z|<\pi,\ a>0)$$

6.1.40

$$\ln\Gamma(z)\sim(z-\tfrac{1}{2})\ln z-z+\tfrac{1}{2}\ln(2\pi)+\sum_{m=1}^{\infty}\frac{B_{2m}}{2m(2m-1)z^{2m-1}}\qquad (z\to\infty \text{ in } |\arg z|<\pi)$$

For B_n see chapter 23

6.1.41

$$\ln\Gamma(z)\sim(z-\tfrac{1}{2})\ln z-z+\tfrac{1}{2}\ln(2\pi)+\frac{1}{12z}-\frac{1}{360z^3}+\frac{1}{1260z^5}-\frac{1}{1680z^7}+\cdots\qquad (z\to\infty \text{ in } |\arg z|<\pi)$$

[3] From C. Hastings, Jr., Approximations for digital computers, Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

Error Term for Asymptotic Expansion

6.1.42

If

$$R_n(z)=\ln\Gamma(z)-(z-\tfrac{1}{2})\ln z+z-\tfrac{1}{2}\ln(2\pi)-\sum_{m=1}^{n}\frac{B_{2m}}{2m(2m-1)z^{2m-1}}$$

then

$$|R_n(z)|\le\frac{|B_{2n+2}|K(z)}{(2n+1)(2n+2)|z|^{2n+1}}$$

where

$$K(z)=\underset{u\ge 0}{\text{upper bound}}|z^2/(u^2+z^2)|$$

For z real and positive, R_n is less in absolute value than the first term neglected and has the same sign.

6.1.43

$$\mathscr{R}\ln\Gamma(iy)=\mathscr{R}\ln\Gamma(-iy)$$
$$=\tfrac{1}{2}\ln\left(\frac{\pi}{y\sinh\pi y}\right)$$
$$\sim\tfrac{1}{2}\ln(2\pi)-\tfrac{1}{2}\pi y-\tfrac{1}{2}\ln y,\qquad (y\to+\infty)$$

6.1.44

$$\mathscr{I}\ln\Gamma(iy)=\arg\Gamma(iy)=-\arg\Gamma(-iy)$$
$$=-\mathscr{I}\ln\Gamma(-iy)$$
$$\sim y\ln y-y-\tfrac{1}{4}\pi-\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_{2n}}{(2n-1)(2n)y^{2n-1}}\qquad (y\to+\infty)$$

6.1.45 $$\lim_{|y|\to\infty}(2\pi)^{-\frac{1}{2}}|\Gamma(x+iy)|e^{\frac{1}{2}\pi|y|}|y|^{\frac{1}{2}-x}=1$$

6.1.46 $$\lim_{n\to\infty}n^{b-a}\frac{\Gamma(n+a)}{\Gamma(n+b)}=1$$

6.1.47

$$z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)}\sim 1+\frac{(a-b)(a+b-1)}{2z}+\frac{1}{12}\binom{a-b}{2}\left(3(a+b-1)^2-a+b-1\right)\frac{1}{z^2}+\ .\ .\ .$$

as $z\to\infty$ along any curve joining $z=0$ and $z=\infty$, providing $z\ne -a,\ -a-1,\ .\ .\ .\ ;\ z\ne -b,\ -b-1,\ .\ .\ .\ .$