### Recurrence Formulas

**6.1.15** $\Gamma(z+1)=z\Gamma(z)=z!=z(z-1)!$

**6.1.16**

$$\Gamma(n+z)=(n-1+z)(n-2+z)\ldots(1+z)\Gamma(1+z)$$
$$=(n-1+z)!$$
$$=(n-1+z)(n-2+z)\ldots(1+z)z!$$

### Reflection Formula

**6.1.17** $\Gamma(z)\Gamma(1-z)=-z\Gamma(-z)\Gamma(z)=\pi\csc\pi z$

$$=\int_0^\infty \frac{t^{z-1}}{1+t}dt \qquad (0<\Re z<1)$$

### Duplication Formula

**6.1.18** $\Gamma(2z)=(2\pi)^{-\frac{1}{2}}2^{2z-\frac{1}{2}}\Gamma(z)\,\Gamma(z+\frac{1}{2})$

### Triplication Formula

**6.1.19** $\Gamma(3z)=(2\pi)^{-1}3^{3z-\frac{1}{2}}\Gamma(z)\Gamma(z+\frac{1}{3})\Gamma(z+\frac{2}{3})$

### Gauss' Multiplication Formula

**6.1.20** $\Gamma(nz)=(2\pi)^{\frac{1}{2}(1-n)}n^{nz-\frac{1}{2}}\prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right)$

### Binomial Coefficient

**6.1.21** $\binom{z}{w}=\frac{z!}{w!(z-w)!}=\frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}$

### Pochhammer's Symbol

**6.1.22**

$$(z)_0=1,$$
$$(z)_n=z(z+1)(z+2)\ldots(z+n-1)=\frac{\Gamma(z+n)}{\Gamma(z)}$$

### Gamma Function in the Complex Plane

**6.1.23** $\Gamma(\bar{z})=\overline{\Gamma(z)};\ \ln\Gamma(\bar{z})=\overline{\ln\Gamma(z)}$

**6.1.24** $\arg\Gamma(z+1)=\arg\Gamma(z)+\arctan\frac{y}{x}$

**6.1.25** $\left|\frac{\Gamma(x+iy)}{\Gamma(x)}\right|^2=\prod_{n=0}^{\infty}\left[1+\frac{y^2}{(x+n)^2}\right]^{-1}$

**6.1.26** $|\Gamma(x+iy)|\leq|\Gamma(x)|$

**6.1.27**

$$\arg\Gamma(x+iy)=y\psi(x)+\sum_{n=0}^{\infty}\left(\frac{y}{x+n}-\arctan\frac{y}{x+n}\right)$$
$$(x+iy\neq 0,-1,-2,\ldots)$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$

**6.1.28** $\Gamma(1+iy)=iy\,\Gamma(iy)$

**6.1.29** $\Gamma(iy)\Gamma(-iy)=|\Gamma(iy)|^2=\frac{\pi}{y\sinh\pi y}$

**6.1.30** $\Gamma(\frac{1}{2}+iy)\Gamma(\frac{1}{2}-iy)=|\Gamma(\frac{1}{2}+iy)|^2=\frac{\pi}{\cosh\pi y}$

**6.1.31** $\Gamma(1+iy)\Gamma(1-iy)=|\Gamma(1+iy)|^2=\frac{\pi y}{\sinh\pi y}$

**6.1.32** $\Gamma(\frac{1}{4}+iy)\Gamma(\frac{3}{4}-iy)=\frac{\pi\sqrt{2}}{\cosh\pi y+i\sinh\pi y}$

### Power Series

**6.1.33**

$$\ln\Gamma(1+z)=-\ln(1+z)+z(1-\gamma)$$
$$+\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]z^n/n \qquad (|z|<2)$$

$\zeta(n)$ is the Riemann Zeta Function (see chapter 23).

### Series Expansion[2] for $1/\Gamma(z)$

**6.1.34** $\frac{1}{\Gamma(z)}=\sum_{k=1}^{\infty}c_kz^k \qquad (|z|<\infty)$

| $k$ | $c_k$ |
|---|---|
| 1 | 1. 00000 00000 000000 |
| 2 | 0. 57721 56649 015329 |
| 3 | —0. 65587 80715 202538 |
| 4 | —0. 04200 26350 340952 |
| 5 | 0. 16653 86113 822915 |
| 6 | —0. 04219 77345 555443 |
| 7 | —0. 00962 19715 278770 |
| 8 | 0. 00721 89432 466630 |
| 9 | —0. 00116 51675 918591 |
| 10 | —0. 00021 52416 741149 |
| 11 | 0. 00012 80502 823882 |
| 12 | —0. 00002 01348 547807 |
| 13 | —0. 00000 12504 934821 |
| 14 | 0. 00000 11330 272320 |
| 15 | —0. 00000 02056 338417 |
| 16 | 0. 00000 00061 160950 |
| 17 | 0. 00000 00050 020075 |
| 18 | —0. 00000 00011 812746 |
| 19 | 0. 00000 00001 043427 |
| 20 | 0. 00000 00000 077823 |
| 21 | —0. 00000 00000 036968 |
| 22 | 0. 00000 00000 005100 |
| 23 | —0. 00000 00000 000206 |
| 24 | —0. 00000 00000 000054 |
| 25 | 0. 00000 00000 000014 |
| 26 | 0. 00000 00000 000001 |

[2] The coefficients $c_k$ are from H. T. Davis, Tables of higher mathematical functions, 2 vols., Principia Press, Bloomington, Ind., 1933, 1935 (with permission); with corrections due to H. E. Salzer.