Example 10. Evaluate the integral $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$ to 4D using the Euler transform.

$$\int_0^\infty \frac{\sin x}{x}\, dx = \sum_{k=0}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{\sin x}{x}\, dx$$

$$= \sum_{k=0}^{\infty} \int_0^{\pi} \frac{\sin (k\pi + t)}{k\pi + t}\, dt = \sum_{k=0}^{\infty} (-1)^k \int_0^{\pi} \frac{\sin t}{k\pi + t}\, dt.$$

Evaluating the integrals in the last sum by numerical integration we get

k	$\int_0^{\pi} \frac{\sin t}{k\pi + t}\, dt$	Δ	Δ^2	Δ^3	Δ^4
0	1.85194				
1	.43379				
2	.25661				
3	.18260				
4	.14180				
		−2587			
5	.11593		799		
		−1788		−321	
6	.09805		478		153
		−1310		−168	
7	.08495		310		
		−1000			
8	.07495				

The sum to $k=3$ is 1.49216. Applying the Euler transform to the remainder we obtain

$$\frac{1}{2}(.14180) - \frac{1}{2^2}(-.02587) + \frac{1}{2^3}(.00799) - \frac{1}{2^4}(-.00321) + \frac{1}{2^5}(.00153)$$

$$= .07090 + .00647 + .00100 + .00020 + .00005$$

$$= .07862$$

We obtain the value of the integral as 1.57078 as compared with 1.57080.

Example 11. Sum the series $\sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6}$ using the Euler-Maclaurin summation formula.

From **3.6.28** we have for $n = \infty$,

$$\sum_{k=1}^{\infty} k^{-2} = \sum_{k=1}^{10} k^{-2} + \sum_{k=1}^{\infty} (k+10)^{-2}$$

$$= \sum_{k=1}^{10} k^{-2} + \int_0^\infty f(k)dk - \frac{1}{2} f_0 - \frac{1}{12} f_0' + \frac{1}{720} f_0''' - \ldots$$

where $f(k) = (k+10)^{-2}$. Thus,

$$\sum_{k=1}^{\infty} k^{-2} = 1.54976\,7731 + .1 - .005 + .00016\,6667 - .00000\,0333$$

$$= 1.64493\,4065,$$

as compared with $\frac{\pi^2}{6} = 1.64493\,4067$.

Example 12. Compute

$$\arctan x = \frac{x}{1+}\,\frac{x^2}{3+}\,\frac{4x^2}{5+}\,\frac{9x^2}{7+}\ \cdots$$

to 5D for $x = .2$. Here $a_1 = x$, $a_n = (n-1)^2 x^2$ for $n > 1$, $b_0 = 0$, $b_n = 2n-1$, $A_{-1} = 1$, $B_{-1} = 0$, $A_0 = 0$, $B_0 = 1$.

For $n \geq 1$

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{vmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{vmatrix} \begin{bmatrix} 2n-1 \\ (n-1)^2 x^2 \end{bmatrix} \qquad \frac{A_0}{B_0} = 0$$

$$\begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 1 \\ .2 \end{vmatrix} = \begin{vmatrix} .2 \\ 1 \end{vmatrix} \qquad \frac{A_1}{B_1} = .2$$

$$\begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{vmatrix} .2 & 0 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} 3 \\ .04 \end{vmatrix} = \begin{vmatrix} .6 \\ 3.04 \end{vmatrix} \qquad \frac{A_2}{B_2} = .197368$$

$$\begin{bmatrix} A_3 \\ B_3 \end{bmatrix} = \begin{vmatrix} .6 & .2 \\ 3.04 & 1 \end{vmatrix} \begin{vmatrix} 5 \\ .16 \end{vmatrix} = \begin{vmatrix} 3.032 \\ 15.36 \end{vmatrix} \qquad \frac{A_3}{B_3} = .197396$$

$$\begin{bmatrix} A_4 \\ B_4 \end{bmatrix} = \begin{vmatrix} 3.032 & .6 \\ 15.36 & 3.04 \end{vmatrix} \begin{vmatrix} 7 \\ .36 \end{vmatrix} = \begin{vmatrix} 21.440 \\ 108.6144 \end{vmatrix} \qquad \frac{A_4}{B_4} = .197396$$

Note that in carrying out the recurrence method for computing continued fractions the numerators A_n and the denominators B_n must be used as originally computed. The numerators and denominators obtained by reducing A_n/B_n to lower terms must not be used.