3.10. Theorems on Continued Fractions

Definitions

3.10.1

(1) Let $$f=b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+}}}\cdots$$

$$=b_0+\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots$$

If the number of terms is finite, f is called a terminating continued fraction. If the number of terms is infinite, f is called an infinite continued fraction and the terminating fraction

$$f_n=\frac{A_n}{B_n}=b_0+\frac{a_1}{b_1+}\frac{a_2}{b_2+}\cdots\frac{a_n}{b_n}$$

is called the nth convergent of f.

(2) If $\lim_{n\to\infty}\frac{A_n}{B_n}$ exists, the infinite continued fraction f is said to be convergent. If $a_i=1$ and the b_i are integers there is always convergence.

Theorems

(1) If a_i and b_i are positive then $f_{2n}<f_{2n+2}$, $f_{2n-1}>f_{2n+1}$.

(2) If $f_n=\frac{A_n}{B_n}$,

$$A_n=b_nA_{n-1}+a_nA_{n-2}$$

$$B_n=b_nB_{n-1}+a_nB_{n-2}$$

where $A_{-1}=1$, $A_0=b_0$, $B_{-1}=0$, $B_0=1$.

(3) $$\begin{bmatrix}A_n\\B_n\end{bmatrix}=\begin{bmatrix}A_{n-1}&A_{n-2}\\B_{n-1}&B_{n-2}\end{bmatrix}\begin{bmatrix}b_n\\a_n\end{bmatrix}$$

(4) $$A_nB_{n-1}-A_{n-1}B_n=(-1)^{n-1}\prod_{k=1}^{n}a_k$$

(5) For every $n\geq 0$,

$$f_n=b_0+\frac{c_1a_1}{c_1b_1+}\frac{c_1c_2a_2}{c_2b_2+}\frac{c_2c_3a_3}{c_3b_3+}\cdots\frac{c_{n-1}c_na_n}{c_nb_n}.$$

(6) $$1+b_2+b_2b_3+\ldots+b_2b_3\ldots b_n$$

$$=\frac{1}{1-}\frac{b_2}{b_2+1-}\frac{b_3}{b_3+1-}\cdots\frac{b_n}{-b_n+1}$$

$$\frac{1}{u_1}+\frac{1}{u_2}+\ldots+\frac{1}{u_n}=\frac{1}{u_1-}\frac{u_1^2}{u_1+u_2-}\cdots\frac{u_{n-1}^2}{-u_{n-1}+u_n}$$

$$\frac{1}{a_0}-\frac{x}{a_0a_1}+\frac{x^2}{a_0a_1a_2}\cdots+(-1)^n\frac{x^n}{a_0a_1a_2\ldots a_n}$$

$$=\frac{1}{a_0+}\frac{a_0x}{a_1-x+}\frac{a_1x}{a_2-x+}\cdots\frac{a_{n-1}x}{+a_n-x}$$

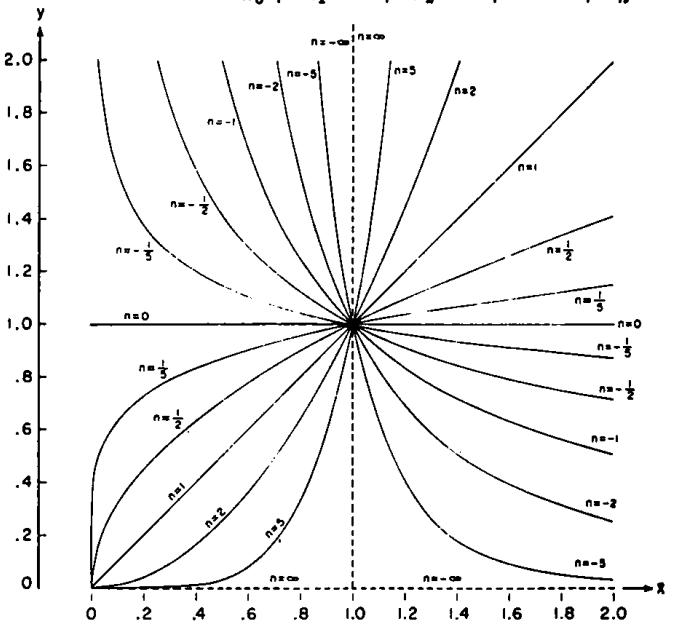

FIGURE 3.1. $y=x^n$.
$\pm n=0, \frac{1}{5}, \frac{1}{2}, 1, 2, 5.$

Numerical Methods

3.11. Use and Extension of the Tables

Example 1. Compute x^{19} and x^{47} for $x=29$ using **Table 3.1.**

$$x^{19}=x^9\cdot x^{10}$$
$$=(1.45071\ 4598\cdot 10^{13})(4.20707\ 2333\cdot 10^{14})$$
$$=6.10326\ 1248\cdot 10^{27}$$
$$x^{47}=(x^{24})^2/x$$
$$=(1.25184\ 9008\cdot 10^{35})^2/29$$
$$=5.40388\ 2547\cdot 10^{68}$$

Example 2. Compute $x^{-3/4}$ for $x=9.19826$.

$$(9.19826)^{1/4}=(919.826/100)^{1/4}=(919.826)^{1/4}/10^{\frac{1}{2}}$$

Linear interpolation in **Table 3.1** gives $(919.826)^{1/4}\approx 5.507144$.

By Newton's method for fourth roots with $N=919.826$,

$$\frac{1}{4}\left[\frac{919.826}{(5.507144)^3}+3(5.507144)\right]=5.50714\ 3845$$

Repetition yields the same result. Thus,

$$x^{1/4}=5.50714\ 3845/10^{\frac{1}{2}}=1.74151\ 1796,$$
$$x^{-3/4}=x^{\frac{1}{2}}/x=.18933\ 05683.$$

3.12. Computing Techniques

Example 3. Solve the quadratic equation $x^2-18.2x+.056$ given the coefficients as $18.2\pm.1$,